1. Fig. 11 shows the curve with parametric equations

$$x = 2\cos\theta,\ y = \sin\theta,\ 0 \le \theta \le 2\pi.$$

The point P has parameter $\frac{1}{4}\pi$. The tangent at P to the curve meets the axes at A and B.

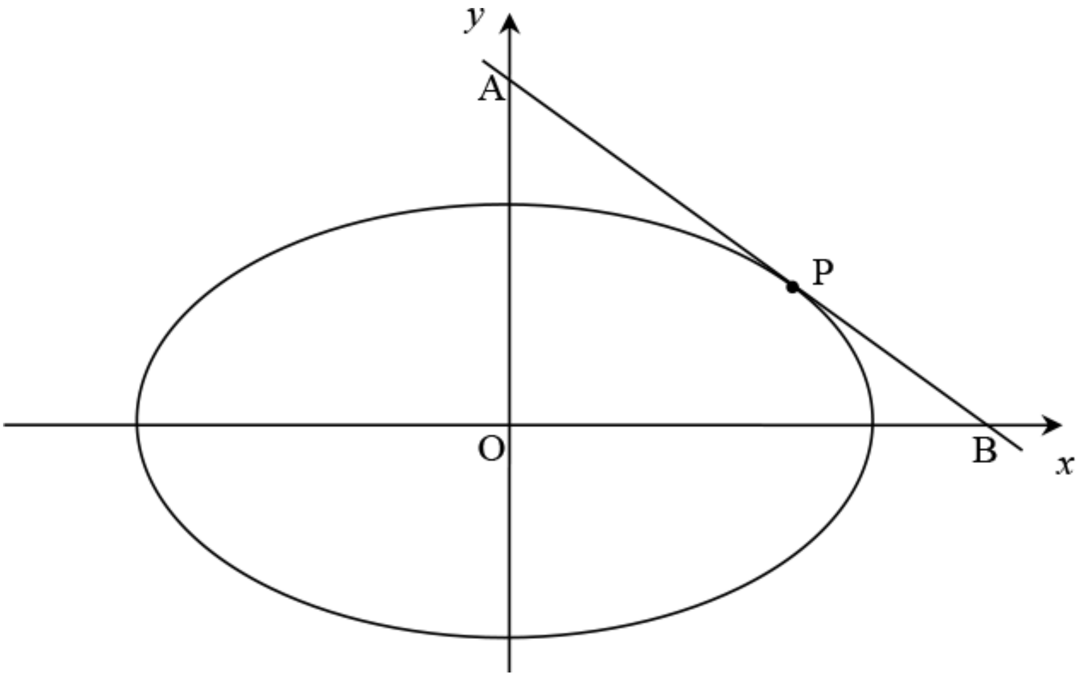

Fig.11

(a) Show that the equation of the line AB is $x + 2y = 2\sqrt{2}$. [6]

(b) Determine the area of the triangle AOB. [3]

2. **In this question you must show detailed reasoning.**

A curve has parametric equations

$$x = \cos t - 3t \text{ and } y = 3t - 4\cos t - \sin 2t, \text{ for } 0 \le t \le \pi.$$

Show that the gradient of the curve is always negative. [7]

3. **In this question you must show detailed reasoning.**

Fig. 8 shows the curve with parametric equations

$$x = 7\cos\theta + 2\cos 2\theta,\ y = 2 + \sin\theta,\ (0 \le \theta \le 2\pi).$$

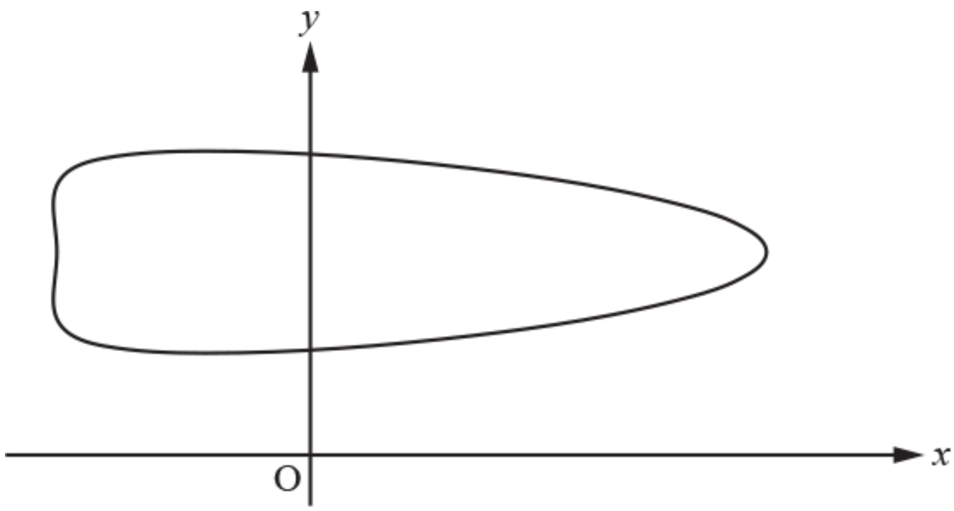

Fig. 8

(a) Find the coordinates of the point on the curve with the greatest y-coordinate. [4]

(b) Determine the exact y-coordinates of the points where the curve crosses the y-axis. [6]

4. A curve has parametric equations $x = \dfrac{t}{1+t^3}$, $y = \dfrac{t^2}{1+t^3}$ where $t \ne -1$.

(a) **In this question you must show detailed reasoning.**

Determine the gradient of the curve at the point where $t = 1$. [5]

(b) Verify that the cartesian equation of the curve is $x^3 + y^3 = xy$. [3]

5. Fig. 9 shows the curve with parametric equations

$$x = \tan\theta,\ y = 1 + \cos 2\theta, \text{ for } 0 \le \theta \le \pi \text{ with } \theta \ne \tfrac{1}{2}\pi.$$

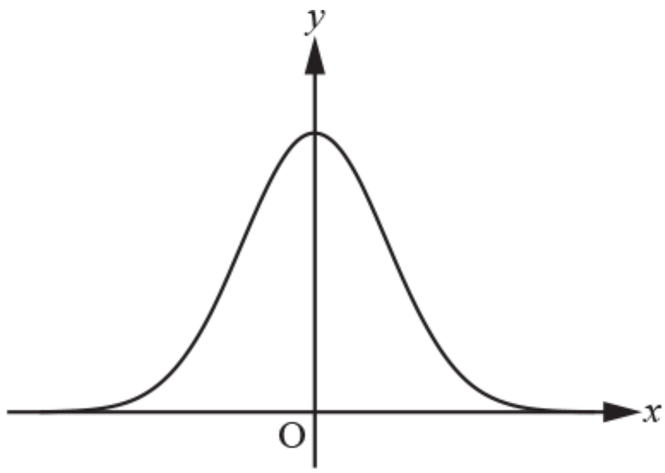

Fig. 9

(a) Explain why $\theta \ne \frac{1}{2}\pi$ [1]

(b) Show that the maximum value of y for points on the curve is 2. [1]

(c) Show that the cartesian equation of the curve is $y = \dfrac{2}{1+x^2}$. [3]

(d) **In this question you must show detailed reasoning.**

The point P in the first quadrant lies on the curve. Find the coordinates of P given that OP is the normal to the curve at P. [7]

6. A curve has parametric equations

$$x = 2 \cot \theta, \qquad y = 2 \sin^2\theta,$$

for values of θ for which both x and y are defined.

(a) For what values of θ, in the interval $0 \le \theta \le 2\pi$, is x undefined? [2]

(b) Show that the cartesian equation of the curve is $y = \dfrac{A}{x^2 + B}$ where A and B are positive integers to be determined. [3]

(c) Ali says that the curve lies completely above the x-axis. Determine whether Ali is correct. [2]

END OF QUESTION paper

Mark scheme

Question			Answer/Indicative content	Marks	Guidance
1		a	P is $(\sqrt{2}, \frac{\sqrt{2}}{2})$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}\theta} \div \frac{\mathrm{d}x}{\mathrm{d}\theta}$ $\boxed{= \frac{\cos\theta}{-2\sin\theta}}$ When $\theta = \frac{\pi}{4}$, $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{2}$ Equation of tangent is $(y - \frac{\sqrt{2}}{2}) = -\frac{1}{2}(x - \sqrt{2})$ $\Rightarrow \quad y = -\frac{1}{2}x + \frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}$ $\Rightarrow \quad x + 2y = 2\sqrt{2}$	B1(AO1.1) M1(AO3.1a) A1(AO1.1) A1(AO1.1) B1(AO2.1) E1(AO1.1) [6]	oe AG
		b	When $x = 0$, y = $\sqrt{2}$ so A is (0, $\sqrt{2}$) When $y = 0$, $x = 2\sqrt{2}$ so B is ($2\sqrt{2}$, 0) Area of triangle $= \frac{1}{2}\sqrt{2} \times 2\sqrt{2} = 2$ units^2	B1(AO1.1) B1(AO1.1) B1(AO1.1) [3]	

			Total	9	
2			DR $\frac{dx}{dt} = -\sin t - 3$ $\frac{dy}{dt} = 3 + 4\sin t - 2\cos 2t$ $\frac{dy}{dx} = \frac{3 + 4\sin t - 2\cos 2t}{-\sin t - 3}$ denominator is always negative since $-1 \le \sin t \le 1$ Substitution of $\cos 2t = 1 - 2\sin^2 t$ in numerator $4\sin^2 t + 4\sin t + 1$ seen in numerator Numerator is $(2\sin t + 1)^2$ so numerator is always positive and denominator is always negative, so $\frac{dy}{dx}$ s always negative	B1(AO 3.1a) B1(AO 1.1) M1(AO 2.1) E1(AO 2.4) M1(AO 1.1) A1(AO 1.1) E1(AO 2.4) [7]	
			Total	7	
3		a	DR $\frac{dy}{dx} = 0$ when $\frac{dy}{d\theta} = \cos\theta = 0$ $\theta = \frac{1}{2}\pi,\ \frac{3}{2}\pi$	M1(AO3.1a) M1(AO1.1)	

			A1(AO1.1)	
		$x = -2$	A1(AO3.2a)	
		$y = 3$	M1	
		Alternative solution	A1	
		Maximum y occurs when $\sin\theta = 1$	M1	
		$y = 3$	A1	
		$\theta = \frac{1}{2}\pi$		
		$x = -2$	[4]	
		DR		
		$7\cos\theta + 2\cos 2\theta = 0 \Rightarrow 4\cos^2\theta + 7\cos\theta - 2 = 0$	M1(AO1.1a)	Use of double angle formula
		$(4\cos\theta - 1)(\cos\theta + 2) = 0$	M1(AO1.1)	Method for solving quadratic
	b	$\cos\theta = \frac{1}{4}$ or -2	A1(AO1.1)	
		Reject $\cos\theta = -2$	B1(AO3.2a)	May be seen later
			M1(AO3.1a)	
		$\sin^2\theta = 1 - \cos^2\theta = 1 - \left(\frac{1}{4}\right)^2 = \frac{15}{16}$	A1(AO2.2a)	

			$y = 2 \pm \frac{\sqrt{15}}{4}$	[6]	oe exact form
			Total	10	
4		a	$\frac{dx}{dt} = \frac{(1+t^3) - t(3t^2)}{(1+t^3)^2}$ $\frac{dx}{dt} = \frac{(1-2t^3)}{(1+t^3)^2}$ oe $\frac{dy}{dt} = \frac{2t(1+t^3) - t^2(3t^2)}{(1+t^3)^2}$ $\frac{dy}{dt} = \frac{(2t - t^4)}{(1+t^3)^2}$ oe	M1 (AO 1.1a) A1 (AO 1.1) A1 (AO 1.1) M1(AO 1.1a) A1(AO 2.1) [5]	DR Use of quotient rule – may be gained for $\frac{dy}{dt}$ if $\frac{dx}{dt}$ not seen(allow ±) Substitution into *their* $\frac{dy}{dx}$ dep on

			$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \div \frac{\mathrm{d}t}{\mathrm{d}x} = \frac{2t - t^4}{1 - 2t^3}$ $t = 1 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -1$		earlier M1 Examiner's Comments This part saw many attempts at the correct process, but there were some mistakes seen in applying the quotient rule generally with confusion over signs. More difficulty was experienced by those who chose to use the product rule but did not appreciate the need to apply the chain rule to the differentiation of$(1 + t^3)^{-1}$. AfI Possibly the use of the product rule to prove the quotient rule leads some candidates to assume that that is the method to use to differentiate quotients. Candidates should know the difference and be able to apply both the product rule and the quotient rule accurately as appropriate.
		b	$\mathrm{LHS} = \frac{t^3 + t^6}{(1 + t^3)^3}$ oe $= \frac{t^3(1 + t^3)}{(1 + t^3)^3}$	M1 (AO 1.1a) M1(AO 1.1) A1 (AO 2.1) [3]	AG Expression for LHS Factorising Completion of argument

			$$\text{RHS} = \frac{t^3}{\left(1+t^3\right)^2} = \text{LHS}$$		<u>Examiner's Comments</u> This part was generally started well, but many candidates struggled to complete the argument to score fully. The majority tackled this question by starting with finding an expression for $x^3 + y^3$ in terms of t, but a few chose to use the expression for x or y and make t the subject then continue to eliminate t from the expressions and generally complete the argument satisfactorily. Exemplar 4

$$x = \frac{t}{1+t^3} \qquad y = \frac{t^2}{1+t^3}$$

$$\frac{1}{y} = \frac{1+t^3}{t^2}$$

$$\frac{x}{y} = \frac{t}{1+t^3} \times \frac{1+t^3}{t^2}$$

$$= \frac{t}{t^2} = t^{-1} = \frac{1}{t}$$

$$t = \frac{y}{x}$$

$$x = \frac{\frac{y}{x}}{1+\left(\frac{y}{x}\right)^3} = x\left(1+\left(\frac{y}{x}\right)^3\right) = \frac{y}{x}$$

$$x + \frac{xy^3}{x^3} = \frac{y}{x}$$

$$x^4 + xy^3 = x^2y$$

~~divide by x~~ $\cancel{x^3 + y^3 = xy}$

$$x^3 + y^3 = xy$$

This candidate does full proof of the result gaining all 3 marks.

Exemplar 5

$$\left(\frac{t}{1+t^3}\right)^3 + \left(\frac{t^2}{1+t^3}\right)^3 = \frac{t^3}{(1+t^3)^3} + \frac{t^6}{(1+t^3)^3}$$

$$= \frac{t^6+t^3}{(1+t^3)^3} = \frac{t^3(t^3+1)}{(1+t^3)^3} = \frac{t^3}{(1+t^3)^2} = xy$$

Because $xy = \left(\frac{t}{1+t^3}\right)\left(\frac{t^2}{1+t^3}\right) = \frac{t^3}{(1+t^3)^2}$

$$\frac{dx}{dt} = (1+t^3)^{-1} - 3t^3(1+t^3)^{-2}$$

$$\frac{dy}{dt} = 2t(1+t^3)^{-1} - 3t^4(1+t^3)^{-2}$$

~~At~~ $t=1$, $\frac{dx}{dt} = -\frac{1}{4}$

$$\frac{dy}{dt} = \frac{1}{4}$$

$$\frac{dy}{dx} = \frac{1}{4} \div -\frac{1}{4} = -1$$

This candidate shows appropriate initial working to verify the equation, but misses out on the final mark, as they do not state what they have shown.

			Total	8	
5		a	x is not defined for this value	E1 (AO 2.4) [1]	
		b	$\cos 2\theta$ has max value 1 when $\theta = 0$ so max y is 2	E1 (AO 2.4) [1]	Or using $\frac{\mathrm{d}y}{\mathrm{d}\theta} = 0$ correctly
		c	$y = 2\cos^2\theta$	M1 (AO 2.2a)	Relevant use of $\cos 2\theta = 2\cos^2\theta - 1$ — At any stage
			$1 + x^2 = \sec^2\theta$	M1 (AO 1.1a)	Relevant use of $\sec 2\theta = 1 + \tan^2\theta$ — At any stage
			So $1+x^2 = \frac{2}{y} \Rightarrow y = \frac{2}{1+x^2}$	E1 (AO 1.1) [3]	**AG** Successful completion — With use of $\sec\theta = \frac{1}{\cos\theta}$

		d	DR $\frac{dy}{dx} = \frac{-4x}{(1+x^2)^2}$ Gradient OP $= \frac{y}{x}$ $\frac{-4x}{(1+x^2)^2} \times \frac{y}{x} = -1$ $\frac{4x}{(1+x^2)^2} \times \frac{2}{x(1+x^2)} = 1 \Rightarrow (1+x^2)^3 = 8 \Rightarrow 1+x^2 = 2$ $x = 1$ Point P is (1, 1)	M1 (AO 1.1) A1 (AO 1.1) B1 (AO 3.1a) M1 (AO 2.2a) M1 (AO 2.1) A1 (AO 2.2a) A1 (AO 3.2a) [7]	Use of chain rule or quotient rule Correct derivative May be implied Form and solve equation for x Allow ±1 at this stage	Or, using parameters: $\frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta} = \frac{-2\sin 2\theta}{\sec^2\theta}$ $\frac{-2\sin 2\theta}{\sec^2\theta} \frac{(1+\cos 2\theta)}{\tan\theta} = -1$ $\frac{4\sin\theta\cos^3\theta}{1} \times \frac{2\cos^3\theta}{\sin\theta} = 1$ $\cos\theta = (\pm)\frac{1}{\sqrt{2}}$

			Total	12	
6		a	$0, 2\pi$ π	B1(AO 1.2) B1(AO 1.2) [2]	
		b	$\operatorname{cosec}^2\theta = 1 + \cot^2\theta$, so $\quad \dfrac{2}{y} = 1 + \left(\dfrac{x}{2}\right)^2$ $\dfrac{2}{y} = \dfrac{4+x^2}{4}$ $y = \dfrac{8}{x^2+4}$	M1(AO 3.1a) M1(AO 1.1) A1(AO 2.2a)	
			Alternative solution 1 $x^2 = 4\cot^2\theta = \dfrac{4\cos^2\theta}{\sin^2\theta}$ $x^2 = \dfrac{2(2-y)}{\frac{1}{2}y}$ $y = \dfrac{8}{x^2+4}$	M1 M1 A1	
			Alternative solution 2 $\dfrac{A}{x^2+B} = \dfrac{A}{4\cot^2\theta + B} = 2\sin^2\theta$	M1 M1	

			$8\cos^2\theta + 2B\sin^2 = A$	A1	
			so $B = 4$ and $A = 8$, giving $y = \dfrac{8}{x^2+4}$	[3]	
		c	$\dfrac{8}{x^2+4} > 0$ for all x	M1(AO 2.1)	**or** $2\sin^2\theta \geq 0$ and when $2\sin^2\theta = 0$, x is undefined
				A1(AO 2.3)	If zero scored, **SC1** for $x^2 + 4$ is never negative oe
			So Ali is correct [because $y > 0$ for all x]	[2]	
			Total	7	