Questions

Q1.

A curve *C* has parametric equations

$$x = 2t - 1, \quad y = 4t - 7 + \frac{3}{t}, \quad t \neq 0$$

Show that the Cartesian equation of the curve *C* can be written in the form

$$y = \frac{2x^2 + ax + b}{x + 1}, \quad x \neq -1$$

where *a* and *b* are integers to be found.

(3)

(Total for question = 3 marks)

Q2.

A curve C has parametric equations

$$x = 3 + 2\sin t, \qquad y = 4 + 2\cos 2t, \qquad 0 \leq t < 2\pi$$

(a) Show that all points on C satisfy $y = 6 - (x - 3)^2$

(2)

(b) (i) Sketch the curve C.

(ii) Explain briefly why C does not include all points of $y = 6 - (x - 3)^2$, $\quad x \in \mathbb{R}$

(3)

The line with equation $x + y = k$, where k is a constant, intersects C at two distinct points.

(c) State the range of values of k, writing your answer in set notation.

(5)

(Total for question = 10 marks)

Q3.

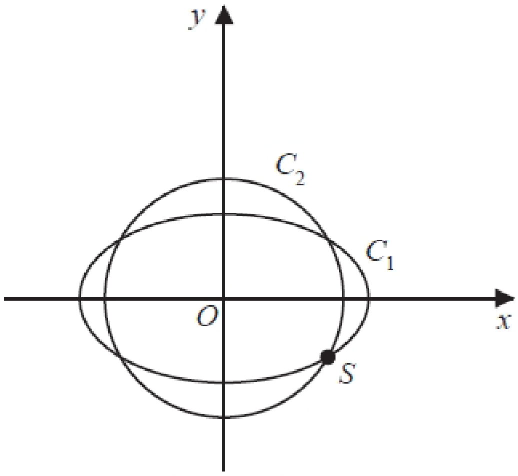


Figure 2

The curve C_1 with parametric equations

$$x = 10\cos t, \quad y = 4\sqrt{2}\sin t, \quad 0 \leqslant t < 2\pi$$

meets the circle C_2 with equation

$$x^2 + y^2 = 66$$

at four distinct points as shown in Figure 2.

Given that one of these points, S, lies in the 4th quadrant, find the Cartesian coordinates of S.

(Total for question = 6 marks)

Q4.

A curve *C* has parametric equations

$$x = \frac{t^2 + 5}{t^2 + 1} \qquad y = \frac{4t}{t^2 + 1} \qquad t \in \mathbb{R}$$

Show that all points on *C* satisfy

$$(x - 3)^2 + y^2 = 4$$

(Total for question = 3 marks)

Q5.

The curve *C* has parametric equations

$x = \sin 2\theta \qquad y = \text{cosec}^3\theta \qquad 0 < \theta < \frac{\pi}{2}$

(a) Find an expression for $\frac{dy}{dx}$ in terms of θ

(3)

(b) Hence find the exact value of the gradient of the tangent to *C* at the point where $y = 8$

(3)

(Total for question = 6 marks)

Q6.

The curve *C* has parametric equations

$$x = 2\cos t, \quad y = \sqrt{3}\cos 2t, \quad 0 \leqslant t \leqslant \pi$$

(a) Find an expression for $\frac{dy}{dx}$ in terms of *t*.

(2)

The point *P* lies on *C* where $t = \frac{2\pi}{3}$

The line *l* is the normal to *C* at *P*.

(b) Show that an equation for *l* is

$$2x - 2\sqrt{3}y - 1 = 0$$

(5)

The line *l* intersects the curve *C* again at the point *Q*.

(c) Find the exact coordinates of *Q*.

You must show clearly how you obtained your answers.

(6)

(Total for question = 13 marks)

Mark Scheme

Q1.

Question	Scheme	Marks	AOs
	Attempts to substitute $=\dfrac{x+1}{2}$ into $y \Rightarrow y = 4\left(\dfrac{x+1}{2}\right)-7 + \dfrac{6}{(x+1)}$	M1	2.1
	Attempts to write as a single fraction $y = \dfrac{(2x-5)(x+1)+6}{(x+1)}$	M1	2.1
	$y = \dfrac{2x^2-3x+1}{x+1} \quad a=-3, b=1$	A1	1.1b
		(3 marks)	

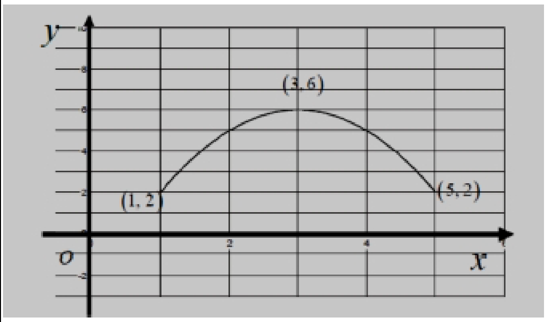
Notes:

M1: Score for an attempt at substituting $t=\dfrac{x+1}{2}$ or equivalent into $y = 4t-7+\dfrac{3}{t}$

M1: Award this for an attempt at a single fraction with a correct common denominator. Their $4\left(\dfrac{x+1}{2}\right)-7$ term may be simplified first

A1: Correct answer only $y=\dfrac{2x^2-3x+1}{x+1} \quad a=-3, b=1$

Q2.

Question	Scheme	Marks	AOs
(a)	Attempts to use $\cos 2t = 1 - 2\sin^2 t \Rightarrow \frac{y-4}{2} = 1 - 2\left(\frac{x-3}{2}\right)^2$	M1	2.1
	$\Rightarrow y - 4 = 2 - 4 \times \frac{(x-3)^2}{4} \Rightarrow y = 6 - (x-3)^2$ *	A1*	1.1b
		(2)	
(b)	 ∩ shaped parabola	M1	1.1b
	Fully correct with 'ends' at (1,2) & (5,2)	A1	1.1b
	Suitable reason : Eg states as $x = 3 + 2\sin t$, $1 \leqslant x \leqslant 5$	B1	2.4
		(3)	

(c)	Either finds the lower value for $k = 7$ or deduces that $k < \frac{37}{4}$	B1	2.2a
	Finds where $x + y = k$ meets $y = 6 - (x-3)^2$ $\Rightarrow k - x = 6 - (x-3)^2$ and proceeds to 3TQ in x or y	M1	3.1a
	Correct 3TQ in x $x^2 - 7x + (k+3) = 0$ Or y $y^2 + (7-2k)y + (k^2 - 6k + 3) = 0$	A1	1.1b
	Uses $b^2 - 4ac = 0 \Rightarrow 49 - 4 \times 1 \times (k+3) = 0 \Rightarrow k = \left(\frac{37}{4}\right)$ or $(7-2k)^2 - 4 \times 1 \times (k^2 - 6k + 3) = 0 \Rightarrow k = \left(\frac{37}{4}\right)$	M1	2.1
	Range of values for $k = \left\{k : 7 \leqslant k < \frac{37}{4}\right\}$	A1	2.5
		(5)	
			(10 marks)

(a)
M1: Uses $\cos 2t = 1-2\sin^2 t$ in an attempt to eliminate t
A1*: Proceeds to $y = 6-(x-3)^2$ without any errors

Allow a proof where they start with $y = 6-(x-3)^2$ and substitute the parametric coordinates. M1 is scored for a correct $\cos 2t = 1-2\sin^2 t$ but A1 is only scored when both sides are seen to be the same AND a comment is made, hence proven, or similar .

(b)
M1: For sketching a $\cap$ parabola with a maximum in quadrant one. It does not need to be symmetrical
A1: For sketching a $\cap$ parabola with a maximum in quadrant one and with end coordinates of $(1,2)$ and $(5,2)$

B1: Any suitable explanation as to why C does not include all points of $y = 6-(x-3)^2,\quad x \in \mathbb{R}$
This should include a reference to **the limits on sin or cos** with a **link to a restriction on** x **or** y.
For example
'As $-1 \le \sin t \le 1$ then $1 \leqslant x \leqslant 5$' Condone in words '$x$ lies between 1 and 5' and strict inequalities
'As $\sin t \le 1$ then $x \leqslant 5$' Condone in words 'x is less than 5'
'As $-1 \le \cos(2t) \le 1$ then $2 \leqslant y \leqslant 6$' Condone in words '$y$ lies between 2 and 6'
Withhold if the statement is incorrect Eg "because the domain is $2 \leqslant x \leqslant 5$"
Do not allow a statement on the top limit of y as this is the same for both curves

(c)
B1: Deduces either

- the correct that the lower value of $k = 7$ This can be found by substituting into $(5,2)$
  $x + y = k \Rightarrow k = 7$ or substituting $x = 5$ into $x^2 - 7x + (k+3) = 0 \Rightarrow 25 - 35 + k + 3 = 0$
  $\Rightarrow k = 7$
- or deduces that $k < \dfrac{37}{4}$ This may be awarded from later work

M1: For an attempt at the upper value for k.
Finds where $x + y = k$ meets $y = 6-(x-3)^2$ once by using an appropriate method.
Eg. Sets $k - x = 6-(x-3)^2$ and proceeds to a 3TQ
A1: Correct 3TQ $x^2 - 7x + (k+3) = 0$ The $= 0$ may be implied by subsequent work
M1: Uses the "discriminant" condition. Accept use of $b^2 = 4ac$ oe or $b^2 \ldots 4ac$ where ... is any inequality leading to a critical value for k. Eg. one root $\Rightarrow 49 - 4\times1\times(k+3) = 0 \Rightarrow k = \dfrac{37}{4}$
A1: Range of values for $k = \left\{k : 7 \leqslant k < \dfrac{37}{4}\right\}$ Accept $k \in \left[7, \dfrac{37}{4}\right)$ or exact equivalent

ALT	As above	B1	2.2a
	Finds where $x + y = k$ meets $y = 6-(x-3)^2$ once by using an appropriate method. Eg. Sets gradient of $y = 6-(x-3)^2$ equal to -1	M1	3.1a
	$-2x + 6 = -1 \Rightarrow x = 3.5$	A1	1.1b
	Finds point of intersection and uses this to find upper value of k. $y = 6-(3.5-3)^2 = 5.75$ Hence using $k = 3.5 + 5.75 = 9.25$	M1	2.1
	Range of values for $k = \{k : 7 \leqslant k < 9.25\}$	A1	2.5

Q3.

Part	Working or answer an examiner might expect to see	Mark	Notes
	$(10\cos t)^2 + (4\sqrt{2}\sin t)^2 = 66$	M1	This mark is given for combining the two equations to show where the curve and circle meet
	$100(\cos t)^2 + 32(1 - \cos t)^2 = 66$	M1	This mark is given for forming an equation in cos t only
	$68\cos^2 t = 34$	A1	This mark is given for simplifying to find an equation in terms of cos t
	$\cos t = \pm\frac{1}{\sqrt{2}} \Rightarrow t = \frac{\pi}{4}$	M1	This mark is given for finding a value for t
	$x = 10 \times \frac{1}{\sqrt{2}}$ $y = 4\sqrt{2} \times -\sin\frac{\pi}{4} = 4\sqrt{2} \times -\frac{1}{\sqrt{2}}$	M1	This mark is given for a method to substitute back into the original equations to find value for x and y
	$S = (5\sqrt{2}, -4)$	A1	This mark is given for the correct coordinates of S
			(Total 6 marks)

Q4.

Question	Scheme	Marks	AOs
	$(x-3)^2+y^2=\left(\frac{t^2+5}{t^2+1}-3\right)^2+\left(\frac{4t}{t^2+1}\right)^2$	M1	3.1a
	$=\frac{(2-2t^2)^2+16t^2}{(t^2+1)^2}=\frac{4+8t^2+4t^4}{(t^2+1)^2}$	dM1	1.1b
	$\frac{4(t^4+2t^2+1)}{(t^2+1)^2}=\frac{4(t^2+1)^2}{(t^2+1)^2}=4*$	A1*	2.1
		(3)	

M1: Attempts to substitute the given parametric forms into the Cartesian equation or the lhs of the Cartesian equation. There may have been an (incorrect) attempt to multiply out the $(x-3)^2$ term.

dM1: Attempts to combine (at least the lhs) using correct processing into a single fraction, multiplies out and collects terms on the numerator.

A1*: Fully correct proof showing all key steps

Question	Scheme	Marks	AOs
Alt	$x=\frac{t^2+5}{t^2+1}\Rightarrow xt^2+x=t^2+5\Rightarrow t^2=\frac{5-x}{x-1}$ $y=\frac{4t}{t^2+1}\Rightarrow y^2=\frac{16t^2}{(t^2+1)^2}=\frac{16\left(\frac{5-x}{x-1}\right)}{\left(\frac{5-x}{x-1}+1\right)^2}$	M1	3.1a
	$y^2=\frac{16\left(\frac{5-x}{x-1}\right)}{\left(\frac{5-x}{x-1}+1\right)^2}=16\left(\frac{5-x}{x-1}\right)\times\left(\frac{(x-1)}{5-x+x-1}\right)^2\Rightarrow y^2=(5-x)(x-1)$	dM1	1.1b
	$y^2=(5-x)(x-1)\Rightarrow y^2=6x-x^2-5$ $\Rightarrow y^2=4-(x-3)^2$ or other intermediate step $\Rightarrow (x-3)^2+y^2=4*$	A1*	2.1
		(3)	
			(3 marks)
	Notes		

M1: Adopts a correct strategy for eliminating t to obtain an equation in terms of x and y only. See scheme.

Other methods exist which also lead to an appropriate equation. E.g using $t=\frac{y}{x-1}$

dM1: Uses correct processing to eliminate the fractions and start to simplify

A1*: Fully correct proof showing all key steps

Q5.

Question	Scheme	Marks	AOs
(a)	$y = \text{cosec}^3\theta \Rightarrow \dfrac{dy}{d\theta} = -3\text{cosec}^2\theta\,\text{cosec}\,\theta\cot\theta$	B1	1.1b
	$\dfrac{dy}{dx} = \dfrac{dy}{d\theta} \div \dfrac{dx}{d\theta}$	M1	1.1b
	$\dfrac{dy}{dx} = \dfrac{-3\text{cosec}^3\theta\cot\theta}{2\cos 2\theta}$	A1	1.1b
		(3)	
(b)	$y = 8 \Rightarrow \text{cosec}^3\theta = 8 \Rightarrow \sin^3\theta = \dfrac{1}{8} \Rightarrow \sin\theta = \dfrac{1}{2}$	M1	3.1a
	$\theta = \dfrac{\pi}{6} \Rightarrow \dfrac{dy}{dx} = \dfrac{-3\text{cosec}^3\left(\frac{\pi}{6}\right)\cot\left(\frac{\pi}{6}\right)}{2\cos\left(\frac{2\pi}{6}\right)} = \ldots$ **or** $\sin\theta = \dfrac{1}{2} \Rightarrow \dfrac{dy}{dx} = \dfrac{\frac{-3}{\sin^3\theta} \times \frac{\cos\theta}{\sin\theta}}{2\left(1-2\sin^2\theta\right)} = \dfrac{-3\times 8 \times \frac{\sqrt{3}/2}{1/2}}{2\left(1-2\times\frac{1}{4}\right)}$	M1	2.1
	$= -24\sqrt{3}$	A1	2.2a
		(3)	
			(6 marks)

Notes
(a)
B1: Correct expression for $\dfrac{dy}{d\theta}$ seen or implied in any form e.g. $\dfrac{-3\cos\theta}{\sin^4\theta}$
M1: Obtains $\dfrac{dx}{d\theta} = k\cos 2\theta$ or $\alpha\cos^2\theta + \beta\sin^2\theta$ (from product rule on $\sin\theta\cos\theta$) and attempts $\dfrac{dy}{dx} = \dfrac{dy}{d\theta} \div \dfrac{dx}{d\theta}$
A1: Correct expression in any form. May see e.g. $\dfrac{-3\cos\theta}{2\sin^4\theta\cos 2\theta}$, $-\dfrac{3}{4\sin^4\theta\cos\theta - 2\sin^3\theta\tan\theta}$
(b)
M1: Recognises the need to find the value of $\sin\theta$ or θ when $y = 8$ and uses the y parameter to establish its value. This should be correct work leading to $\sin\theta = \dfrac{1}{2}$ or e.g. $\theta = \dfrac{\pi}{6}$ or 30°.
M1: Uses their value of $\sin\theta$ or θ in their $\dfrac{dy}{dx}$ from part (a) (working in exact form) in an attempt to obtain an exact value for $\dfrac{dy}{dx}$. May be implied by a correct exact answer.
If no working is shown but an exact answer is given you may need to check that this follows their $\dfrac{dy}{dx}$.
A1: Deduces the correct gradient

Q6.

Question	Scheme	Marks	AOs
(a)	Attempts $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$	M1	1.1b
	$\frac{dy}{dx} = \frac{\sqrt{3}\sin 2t}{\sin t} \quad (= 2\sqrt{3}\cos t)$	A1	1.1b
		(2)	
(b)	Substitutes $t = \frac{2\pi}{3}$ in $\frac{dy}{dx} = \frac{\sqrt{3}\sin 2t}{\sin t} = (-\sqrt{3})$	M1	2.1
	Uses gradient of normal $= -\frac{1}{dy/dx} = \left(\frac{1}{\sqrt{3}}\right)$	M1	2.1
	Coordinates of $P = \left(-1, -\frac{\sqrt{3}}{2}\right)$	B1	1.1b
	Correct form of normal $y + \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{3}}(x+1)$	M1	2.1
	Completes proof $\Rightarrow 2x - 2\sqrt{3}y - 1 = 0$ *	A1*	1.1b
		(5)	
(c)	Substitutes $x = 2\cos t$ and $y = \sqrt{3}\cos 2t$ into $2x - 2\sqrt{3}y - 1 = 0$	M1	3.1a
	Uses the identity $\cos 2t = 2\cos^2 t - 1$ to produce a quadratic in $\cos t$	M1	3.1a
	$\Rightarrow 12\cos^2 t - 4\cos t - 5 = 0$	A1	1.1b
	Finds $\cos t = \frac{5}{6}, \cancel{-\frac{1}{2}}$	M1	2.4
	Substitutes their $\cos t = \frac{5}{6}$ into $x = 2\cos t$, $y = \sqrt{3}\cos 2t$,	M1	1.1b
	$Q = \left(\frac{5}{3}, \frac{7}{18}\sqrt{3}\right)$	A1	1.1b
		(6)	
			(13 marks)

Notes:

(a)

M1: Attempts $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ and achieves a form $k\frac{\sin 2t}{\sin t}$ Alternatively candidates may apply the double angle identity for $\cos 2t$ and achieve a form $k\frac{\sin t \cos t}{\sin t}$

A1: Scored for a correct answer, either $\frac{\sqrt{3}\sin 2t}{\sin t}$ or $2\sqrt{3}\cos t$

(b)

M1: For substituting $t = \frac{2\pi}{3}$ in their $\frac{dy}{dx}$ which must be in terms of t

M1: Uses the gradient of the normal is the negative reciprocal of the value of $\frac{dy}{dx}$. This may be seen in the equation of l.

B1: States or uses (in their tangent or normal) that $P = \left(-1, -\frac{\sqrt{3}}{2}\right)$

M1: Uses their numerical value of $-1\big/\frac{dy}{dx}$ with their $\left(-1, -\frac{\sqrt{3}}{2}\right)$ to form an equation of the normal at P

A1*: This is a proof and all aspects need to be correct. Correct answer only $2x - 2\sqrt{3}y - 1 = 0$

(c)

M1: For substituting $x = 2\cos t$ and $y = \sqrt{3}\cos 2t$ into $2x - 2\sqrt{3}y - 1 = 0$ to produce an equation in t. Alternatively candidates could use $\cos 2t = 2\cos^2 t - 1$ to set up an equation of the form $y = Ax^2 + B$.

M1: Uses the identity $\cos 2t = 2\cos^2 t - 1$ to produce a quadratic equation in $\cos t$
In the alternative method it is for combining their $y = Ax^2 + B$ with $2x - 2\sqrt{3}y - 1 = 0$ to get an equation in just one variable

A1: For the correct quadratic equation $12\cos^2 t - 4\cos t - 5 = 0$
Alternatively the equations in x and y are $3x^2 - 2x - 5 = 0$ $12\sqrt{3}y^2 + 4y - 7\sqrt{3} = 0$

M1: Solves the quadratic equation in $\cos t$ (or x or y) and rejects the value corresponding to P.

M1: Substitutes their $\cos t = \frac{5}{6}$ or their $t = \arccos\left(\frac{5}{6}\right)$ in $x = 2\cos t$ and $y = \sqrt{3}\cos 2t$
If a value of x or y has been found it is for finding the other coordinate.

A1: $Q = \left(\frac{5}{3}, \frac{7}{18}\sqrt{3}\right)$. Allow $x = \frac{5}{3}, y = \frac{7}{18}\sqrt{3}$ but do not allow decimal equivalents.