Parametric Equations Cheat Sheet

So far, we have only looked at functions given in two variables, y and x. This is known as the cartesian equation of a curve. We can also define a curve using a different system, known as parametric equations.

We define the x and y coordinates separately, in terms of a third variable, t:

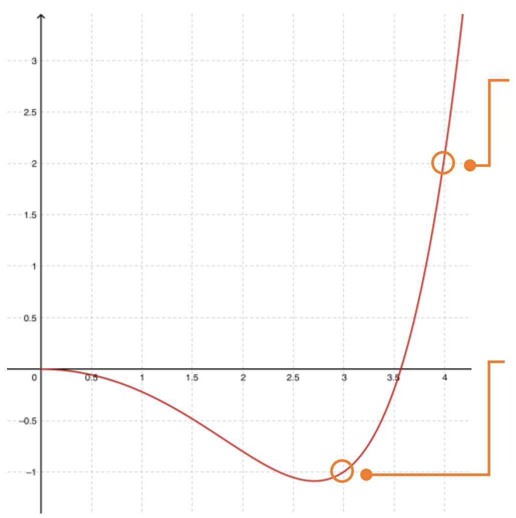
- $x = p(t)$
- $y = q(t)$

Each value of t defines a point on the curve.

To develop a better understanding of how this works, let's look at the following curve defined parametrically:

$x = 3\sqrt{t}, \quad y = t^3 - 2t, \quad 0 < t < 2$

This is the domain of the parameter t, which tells us the values of t which we can substitute into x and y to find points on the curve.



If we wanted to find the value of t at this point (4, 2) then using $x = 3\sqrt{t}$, we let $x = 4$: $4 = 3\sqrt{t}$, $\frac{4}{3} = \sqrt{t} \therefore t = \frac{16}{9}$

We could also let $y = 2$ and solve $2 = t^3 - 2t$, but this would be more difficult.

At $t = 1$, we have that $x = 3\sqrt{1} = 3$ and $y = (1)^2 - 2(1) = -1$
$\therefore$ the point $(3, -1)$ corresponds to when $t = 1$.

Converting between parametric and cartesian equations

To convert between parametric and cartesian equations, you must use substitution to eliminate the parameter. You also need to be able to relate the domain and range of a cartesian equation to its parametric counterpart. Remember that:

- The domain of $f(x)$ is the range of $p(t)$
- The range of $f(x)$ is the range of $q(t)$

Example 1: A curve has parametric equations. $x = \ln(4 - t), \quad y = t - 2, \quad t < 3$
(a) Find the cartesian equation for the curve in the form $y = f(x)$.
(b) Find the domain and range of $f(x)$.

a) Using $x = \ln(4 - t)$, we start by making t the subject:	$e^x = 4 - t$ $\therefore \quad t = 4 - e^x$
Substituting into y:	$y = (4 - e^x) - 2$ $\Rightarrow y = 2 - e^x$
b) We use the domain/range properties of parametric functions to deduce the domain and range of $f(x)$	The domain of $f(x)$ is the range of $\ln(4 - t)$ for $t < 3$. By a sketch or otherwise, you can deduce this is $x > 0$. The range of $f(x)$ is the range of $t - 2$ for $t < 3$. This will be $y < 1$.

When the parametric equations involve trigonometric functions, you may need to use trigonometric identities to convert to cartesian form. Here are two examples showing how this is done in practice:

Example 2: A curve C has parametric equations
$x = \cot t, \quad y = cosec^2 t - 2, \quad 0 < t < \pi$

Find the cartesian equation of the curve in the form $y = f(x)$.

Using $1 + cot^2 x = cosec^2 x$	$x^2 = cot^2 t = cosec^2 t - 1$ $so\ x^2 + 1 = cosec^2 t$
Substituting into y:	$\Rightarrow y = x^2 + 1 - 2$ $\Rightarrow y = x^2 - 1$
Using $1 + cot^2 x = cosec^2 x$	$x^2 = cot^2 t = cosec^2 t - 1$ $so\ x^2 + 1 = cosec^2 t$

Example 3: A curve C has parametric equations
$x = 2cost, \qquad y = \sin\left(t - \frac{\pi}{6}\right), \qquad 0 < t < \pi$
Find a cartesian equation of the curve in the form $y = f(x)$, stating its domain.

a) We start by expanding y using the addition formulae from Chapter 7 of Pure Year 2:	$y = \sin(t)\, cos\left(\frac{\pi}{6}\right) - \cos(t)\, sin\left(\frac{\pi}{6}\right)$ $= \frac{\sqrt{3}}{2} sint - \frac{1}{2} cost$
Using the result from the previous step and substituting $x = 2cost$:	Since $x = 2cost$, $y = \frac{\sqrt{3}}{2} sint - \frac{1}{4} x$.
Now we need to substitute out $sint$ as it is the only remaining term with t in it. We can use the identity $sin^2 t + cos^2 t = 1$ to do this.	$\left(\frac{x}{2}\right) = cost \therefore \left(\frac{x}{2}\right)^2 = cos^2 t$ So $sin^2 t = 1 - \left(\frac{x}{2}\right)^2$ and $sint = \sqrt{1 - \left(\frac{x}{2}\right)^2}$
Substituting this new expression for $sint$ back into the expression from the 2nd step: We now have the required cartesian equation.	$\therefore y = \frac{\sqrt{3}}{2}\left(\sqrt{1 - \left(\frac{x}{2}\right)^2}\right) - \frac{1}{4} x$
We look at the range of the given parametric equation, $x = 2cost$, to find the domain of our cartesian equation. Alternatively, you could also substitute $t = 0$ and $t = \pi$ into $x = 2cost$ to find the domain.	The domain of this function is the range of $x = 2cost$ for $0 < t < \pi$. By a quick sketch of $x = 2cost$, we can see that x takes on all values between but not including -2 and 2, so our is $-2 < x < 2$

Sketching parametric equations

Parametric curves are usually more difficult to sketch than curves given in cartesian form.

- To plot parametric curves, we need to construct a table of values and use it to sketch the curve.

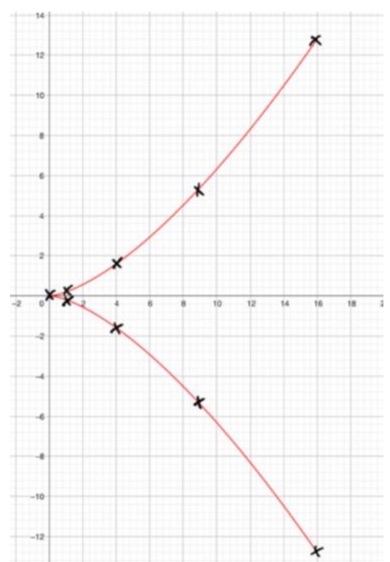
Example 4: A curve is given by the parametric equations $x = t^2$, $y = \frac{t^3}{5}$
Sketch the curve for $-4 \le t \le 4$.

We start by constructing the following table and filling it in:

t	-4	-3	-2	-1	0	1	2	3	4
$x = t^2$	16	9	4	1	0	1	4	9	16
$y = \frac{t^3}{5}$	-12.8	-5.4	-1.6	-0.2	0	0.2	1.6	5.4	12.8

Note that you can use as many or as little t values in your table as you like. In this case using the 9 integers in the range $-4 \le t \le 4$ is enough for us to figure out the shape. You can always use more values if you feel the need.

Plotting our points then sketching the curve that goes through all of them:



Points of intersection

Further problems will involve the use of coordinate geometry. You will often need to find intersections between curves defined parametrically and functions given in cartesian form.

With such questions, the general procedure is to substitute your parametric equations into your cartesian equation, resulting in an equation for t which should be solved. The solutions to this equation represent the values of t where the two functions intersect.

Example 5: Find the points of intersection of the parabola $x = t^2, y = 2t$ with the circle
$x^2 + y^2 - 9x + 4 = 0.$

Substituting $x = t^2, y = 2t$ into the circle:	$(t^2)^2 + (2t)^2 - 9(t^2) + 4 = 0$ $\Rightarrow t^4 + 4t^2 - 9t^2 + 4 = 0$ $\Rightarrow t^4 - 5t^2 + 4 = 0$
Solving the quadratic:	The solutions to this equation via the quadratic formula are $t =$ -1, 1, 2, or -2
To find the points, we need to substitute these values of t back into the given parameterisation $x = t^2, y = 2t$. Doing so, starting with $t =$ -1 and 1 :	$x = 1 \quad x = 1$ $y = -2 \quad y = 2$
Now with $t = 2$ and -2	$x = 4 \quad x = 4$ $y = 4 \quad y = -4$
Writing our solutions as coordinates:	$\therefore$ our points are (1,2) , (1,-2), (4,-4) and (4,4)

Modelling with parametric equations

You need to be able to use your knowledge of parametric equations to solve problems involving real-life scenarios. The mathematical techniques used for such problems are no different to regular questions, but in order to succeed you need to make sure you fully understand the scenario given in the question, so take some time to read through the question properly.

Mechanics problems are a popular choice for modelling questions.

Example 6: The path of a skateboarder from the point of leaving a ramp to the point of landing is modelled using the parametric equations

$x = 25t, \quad y = -4.9t^2 + 4t + 15, \quad 0 \le t \le k$

where x is the horizontal distance in meters from the point of leaving the ramp and y is the height in metres above ground level of the skateboarder, after t seconds.

a) Find the initial height of the skateboarder.
b) Find the value of k and hence state the time taken for the skateboarder to complete his jump.
c) Find the horizontal distance the skateboarder jumps.
d) Show that the skateboarder's path is a parabola according to the given model and find the maximum height above ground level of the skateboarder.

a) The initial height is the value of y at $t = 0$.	$\Rightarrow y = 15$
b) k is the value of t when the skateboarder finally lands. This is when $y = 0$. Solving $y = 0$:	$-4.9t + 4t^2 + 15 = 0 \Rightarrow t = 2.205, t = -1.388$ But since t represents time, it cannot be negative. So time taken $= 2.21\ s$ to 3 significant figures.
c) As x represents horizontal distance, we simply need to find x at $t = k = 2.21$	$x = 25(2.205\ldots) = 55.1$ to 3 significant figures.
d) We need to convert into the cartesian form. Finding t in terms of x:	$x = 25t \therefore t = \frac{x}{25}$
Substituting into y:	So $y = -4.9\left(\frac{x}{25}\right)^2 + 4\left(\frac{x}{25}\right) + 15$ $\Rightarrow y = -\frac{49}{6250}x^2 + \frac{4}{25}x + 15$
This is the equation of a parabola, which shows that the skateboarder's path is a parabola (according to the given model). The maximum height of the skateboarder will be the maximum value of y. Differentiating y and equating to 0:	$\frac{dy}{dx} = -\frac{98}{6250}x + \frac{4}{25} = 0$
Solving for x: $y = \frac{6250}{98} \times \frac{4}{25}$:	$\Rightarrow y = \frac{500}{49}$