Circles Cheat Sheet

Midpoints and perpendicular bisector

The midpoint of a line segment can be calculated in a similar fashion to calculating averages of two numbers using the formula

- Midpoint $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Where (x_1, y_1) and (x_2, y_2) are the end points of the line segment.

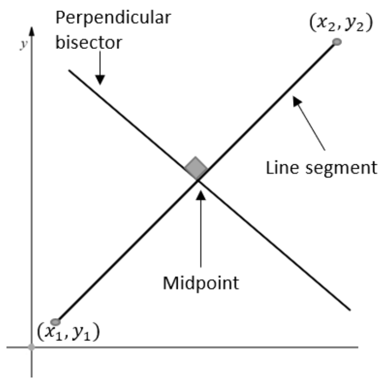
- A perpendicular bisector of a line segment is a line that is perpendicular to the line segment and passes through the midpoint of the line segment.

Example 1: The line segment PQ is the diameter of a circle, where P is at (2,3) and Q is at (6,7). Find the coordinates of the midpoint of the circle.

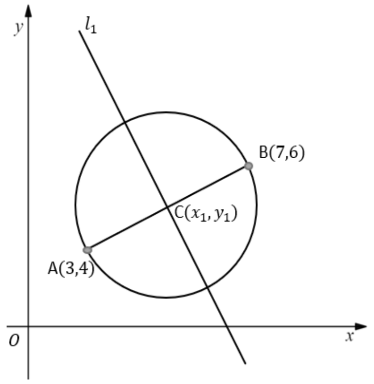
As line segment PQ is the diameter of the circle, the midpoint of the line is the centre of the circle.

Midpoint of PQ: $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{2+6}{2}, \frac{3+7}{2}\right) = (4,5)$

Hence, the centre of the circle is (4,5).



Example 2: Line l_1 passes through the centre of the circle (x_1, y_1) and is perpendicular to the line segment AB. Find the centre of the circle and the equation of the line l_1.



As line segment AB is the diameter of the circle, the midpoint of the line is the centre of the circle.

Midpoint of AB: $\left(\frac{3+7}{2}, \frac{6+4}{2}\right) = (5,5)$

Hence, the centre of the circle is C (5,5).

As line l_1 is perpendicular to the line segment, if the gradient of the line segment is m_1, then the gradient of l_1 is $m_2 = -\frac{1}{m_1}$

Gradient of the line segment AB:

$$m_1 = \frac{6-4}{7-3} = \frac{2}{4} = \frac{1}{2}$$

Hence, the gradient of the line l_1 is

$$m_2 = -\frac{1}{\frac{1}{2}} = -2$$

The line l_1 passes through the centre C(5,5). Therefore, the equation of the line l_1 is

$$y - 5 = -2(x-5)$$
$$y - 5 = -2x + 10$$
$$y = -2x + 15$$

Equation of a circle

- The equation of any given circle with centre (a, b) and radius r is

$$(x-a)^2 + (y-b)^2 = r^2$$

As all points on the circumference have the same distance to its centre, the radius, the equation of the circle is derived using Pythagoras' theorem.

You may come across circles that have equations that come in the form of

- $ax^2 + by^2 + cx + dy + e = 0$

The equations that come in this form are just $(x-a)^2 + (y-b)^2 = r^2$ multiplied out and simplified. To get from the form $ax^2 + by^2 + cx + dy + e = 0$ to $(x-a)^2 + (y-b)^2 = r^2$. You can use the method of completing the square.

Example 3: Find the centre and radius of the circle with equation

$$x^2 + y^2 - 4x - 6y - 3 = 0$$

First, we need to rearrange the equation, so all the like terms are together:

$$\underbrace{x^2 - 4x} + \underbrace{y^2 - 6y} = 3$$

We need to complete the square of $x^2 - 4x$ and $y^2 - 6y$.

$$x^2 - 4x = (x-2)^2 - 4$$
$$y^2 - 6y = (y-3)^2 - 9$$

Now we need to substitute this into the equation of the circle

$$(x-2)^2 - 4 + (y-3)^2 - 9 = 3$$
$$(x-2)^2 + (y-3)^2 = 16$$

Hence, the centre of the circle is at (2,3) and radius of the circle is $r = \sqrt{16}$=4

Intersections of straight lines and circles

You can use similar methods to find intersection points of a straight line and a circle as you did with a straight line and another straight line or a curve.

There can either be one, two or zero intersection points between a straight line and a circle.

Example 4: The line l_1 $y = 2x + 3$ meets the circle $(x-3)^2 + (y-4)^2 = 9$ at two distinct points. Find the x-coordinates of these two intersection points.

In order to find the solution, we can solve the equations simultaneously.

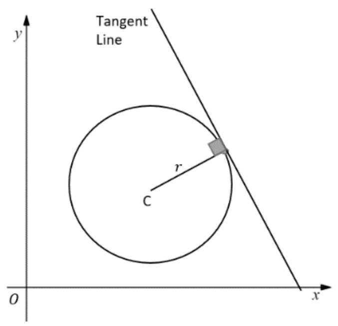
$$(x-3)^2 + (y-4)^2 = 9$$
$$y = 2x + 3$$

$$(x-3)^2 + (2x+3-4)^2 = 9$$
$$(x-3)^2 + (2x-1)^2 = 9$$
$$x^2 - 6x + 9 + 4x^2 - 4x + 1 = 9$$
$$5x^2 - 10x + 1 = 0$$

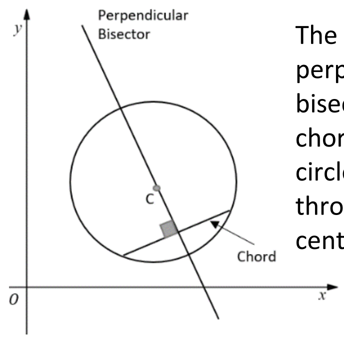
We can use the quadratic equation to solve this.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-(-10) \pm \sqrt{(-10)^2 - 4 \times 5 \times 1}}{2 \times 5}$$
$$x = \frac{5 + 2\sqrt{5}}{5} \quad and \ x = \frac{5 - 2\sqrt{5}}{5}$$

Use tangent and chord properties

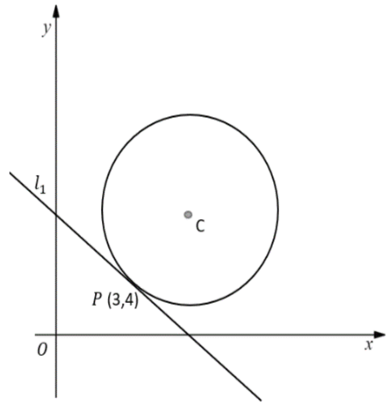


The tangent to the circle is perpendicular to the radius of the circle at the intersection point.



The perpendicular bisector to a chord of the circle will pass through the centre C.

Example 5: The circle has equation $x^2 + y^2 - 10x - 10y + 45 = 0$. The line l_1 is tangent to the circle at point P. Find the equation of this tangent l_1.



The circle has equation $x^2 + y^2 - 10x - 10y + 45 = 0$. To find the centre point, we need to use the completing the square method to get the equation into the form of

$$(x-a)^2 + (y-b)^2 = r^2$$
$$(x-5)^2 - 25 + (y-5)^2 - 25 + 45 = 0$$
$$(x-5)^2 + (y-5)^2 = 5$$

Hence, the centre is at $C(5,5)$. We can now work out the gradient of CP.

$$m_1 = \frac{5-4}{5-3} = \frac{1}{2}$$

The tangent property states that the tangent to the circle is perpendicular to the radius of the circle at the intersection point. Hence line l_1 is perpendicular to CP. Therefore, the gradient of l_1 is

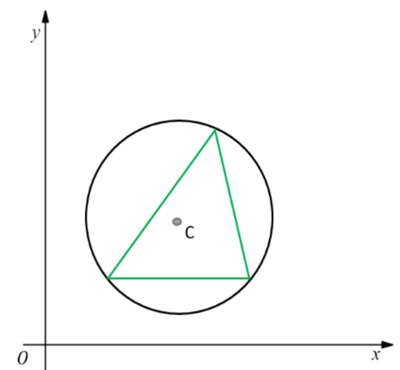
$$m_2 = -\frac{1}{\frac{1}{2}} = -2$$

We know the gradient of l_1, $m_2 = -2$, and a point that the line passes trough $P(3,4)$. The equation of line l_1 can be found from this.

$$y - y_1 = m(x - x_1)$$
$$y - 4 = -2(x-3)$$
$$y - 4 = -2x + 6$$
$$y = -2x + 10$$

Circles and triangles

- A circle that passes through the vertices of a triangle is called a circumcircle of the triangle. The perpendicular bisectors of each side of the triangle intersect at the centre of the circle. This point is called the circumcentre.
- If the triangle is a right-angled triangle., the hypotenuse of the triangle would be the diameter of the circle.
- The angle in a semi-circle will always be 90^o



You can find the centre of a circle using any three points on the circumference. The intersection point of the perpendicular bisector of any two chords formed by the three points on the circumference is the centre of the circle. Hence, you need to find the equation of the respective perpendicular bisectors and find this point of intersection.