1. The functions f and g are defined for all real values of x by

$$f(x) = x^2 + 4ax + a^2 \text{ and } g(x) = 4x - 2a,$$

where a is a positive constant.

i. Find the range of f in terms of a.

[4]

ii. Given that fg(3) = 69, find the value of a and hence find the value of x such that $g^{-1}(x) = x$.

[6]

2. The functions f and g are defined as follows:

$$f(x) = 2 + \ln(x + 3) \text{ for } x \geqslant 0,$$

$g(x)\ ax^2$ for all real values of x, where a is a positive constant.

i. Given that $gf(e^4 - 3) = 9$, find the value of a.

[3]

ii. Find an expression for $f^{-1}(x)$ and state the domain of f^{-1}.

[3]

iii. Given that $ff(e^N - 3) = \ln(53e^2)$, find the value of N.

[5]

3. The functions f and g are defined for all real values of x by

$$f(x) = |\,2x + a\,| + 3a \text{ and } g(x) = 5x - 4a,$$

where a is a positive constant.

i. State the range of f and the range of g.

[2]

ii. State why f has no inverse, and find an expression for $g^{-1}(x)$.

[3]

iii. Solve for x the equation $gf(x) = 31a$.

[5]

4. The function f is defined as $f(x) = \dfrac{8}{x+2}$ for $x \geqslant 0$.

(a) State the range of f. [1]

(b) Find an expression for $f^{-1}(x)$. [2]

(c) Solve the equation $f(x) = f^{-1}(x)$. [2]

5. A sequence of three transformations maps the curve $y = \ln x$ to the curve $y = e^{3x} - 5$.
Give details of these transformations.

[4]

6. The function f is defined for all real values of x as $f(x) = c + 8x - x^2$, where c is a constant.

(a) Given that the range of f is $f(x) \leq 19$, find the value of c. [3]

(b) Given instead that $ff(2) = 8$, find the possible values of c. [4]

7. **In this question you must show detailed reasoning.**

The functions f and g are defined for all real values of x by

$$f(x) = x^3 \text{ and } g(x) = x^2 + 2.$$

(a) Write down expressions for

(i) $fg(x)$, [1]

(ii) $gf(x)$. [1]

(b) Hence find the values of x for which $fg(x) - gf(x) = 24$. [6]

END OF QUESTION paper

Mark scheme

Question			Answer/Indicative content	Marks	Guidance
1		i	Attempt completion of square at least as far as $(x + 2a)^2$ or differentiation to find stationary point at least as far as linear equation involving two terms	*M1	or equiv but a must be present
		i	Obtain $(x + 2a)^2 - 3a^2$ or $(-2a, -3a^2)$	A1	
		i	Attempt inequality involving appropriate y-value	M1	dep *M; allow <, > or ≤ here; allow use of x; or unsimplified equiv
		i	State $y \geq -3a^2$ or $f(x) \geq -3a^2$	A1	now with ≥; here $x \geq -3a^2$ is A0 **Examiner's Comments** This apparently straightforward request was not answered well and 52% of candidates scored no marks. Whether the poor response was due to lack of knowledge of range or to the presence of the indeterminate constant a was not clear. Some candidates attempted to complete the square but this was not always done well and many did not know how to deduce the range from their version. The other popular approach involved differentiation; again candidates were not sure how to conclude and it was common for differentiation to lead to $x = -2a$ with a consequent statement that the range was $x > -2a$. An error that was seen many times in attempts to find the stationary point was $f'(x) = 2x + 4a + 2a$.
		ii	Attempt composition of f and g the right way round	*M1	algebraic or (part) numerical; need to see $4x - 2a$ replacing x at least once
		ii	Obtain or imply $16x^2 - 3a^2$ or $144 - 3a^2$	A1	or less simplified equiv but with at least the brackets expanded correctly
		ii	Attempt to find a from fg(3) = 69	M1	dep *M
		ii	Obtain at least $a = 5$	A1	
		ii	Attempt to solve $4x - 10 = x$ $\frac{1}{4}(x+10) = x$ or $4x - 10 = \frac{1}{4}(x+10)$	M1	for their a; must be linear equation in one variable; condone sign slip in finding inverse of g
		ii	Obtain $\frac{10}{3}$	A1	and no other answer **Examiner's Comments** In contrast to part (i), this part was answered well with 45% of candidates earning all six marks. Many more earned four or five marks, only faltering towards the end of their solutions. The vast majority formed the composition of the functions the right way round and generally coped well with the algebraic simplification. There were some slips and a few equated fg(3) to 0 rather than to 69 but most reached $a^2 = 25$ without too much trouble. There was no penalty at this stage for stating $a = \pm 5$. neat use of $f(x)$ in its completed square form to produce F $g(x) = f(4x - 2a) = (4x - 2a + 2a)^2 - 3a^2 = 16x^2 - 3a^2$ was

					noted a few times. Somewhat strangely, after finding a, a significant number of candidates did no more on this question or found an expression for $g^{-1}(x)$ but then did not consider the equation $g^{-1}(x) = x$. Of course, it is not necessary to find the inverse of g; solving the equation $g(x) = x$ is an equivalent, and easier, process as a few alert candidates recognised.
			Total	10	
2		i	Obtain 6 or 2 + 4 at any stage for application off	B1	
		i	Attempt composition of functions the right way round	M1	
		i	Obtain $a = \frac{1}{4}$ or $\frac{9}{36}$ or equiv	A1	**Examiner's Comments** Some of the marks in this question were easily earned but there were other more demanding aspects. Only 12% of the candidates managed to earn all eleven marks. Part (i) was answered well, particularly by those candidates who simplified $f(e^4 - 3)$ as their first step. Those who formed $gf(x)$ and attempted to expand $a[2 + \ln(x + 3)]^2$ before any substitution for x had more scope for error. Almost all candidates recognised the composition of functions and applied f and g in the correct order. A moment's carelessness led a surprising number of candidates to follow with . $36a = 9$ with $a = 4$.
		ii	Obtain expression involving e^{y-2} or e^{x-2}	M1	
		ii	Obtain $e^{x-2} - 3$	A1	
		ii	State $x \geq 2 + \ln 3$ or equiv	B1	Not for >; not for decimal equiv; using x **Examiner's Comments** The vast majority of candidates had no difficulty in finding $f^{-1}(x)$ but few were successful in stating the correct domain. Common responses were all real numbers, $x \geq 0$ and $x \geq -3$. Those candidates who recognised the link between the range of f and the domain of its inverse usually provided the correct answer.
		iii	Either:		
		iii	Apply f once to obtain $2 + N$	B1	
		iii	Apply f to their expression involving N	M1	
		iii	Obtain $2 + \ln(N + 5)$ or $2 + \ln(2 + N + 3)$	A1	
		iii	Attempt solution of equation of form $2 + \ln(pN + q) = \ln(53e^2)$	M1	Involving manipulation so that value of N is apparent
		iii	Obtain 48 from correct work	A1	
		iii	Or 1:		
		iii	Obtain $ff(x)$ of form $k_1 + \ln[k_2 + \ln(x + 3)]$	M1	Or equiv with immediate substitution for x;

		iii	Obtain correct 2 + ln[5 + ln(x + 3)]	A1	missing bracket(s) may be implied by
		iii	Substitute for x to obtain 2 + ln(N + 5)	A1	subsequent work
		iii	Attempt solution of equation of form 2 + ln(pN + q) = ln(53e^2)	M1	Involving manipulation so that value of N is apparent **Examiner's Comments** Part (iii) was more challenging and 43% of the candidates earned all five marks. Different approaches were seen. Some started by forming ff(x) before substituting e^N – 3 whereas another common method involved finding and simplifying f(e^N – 3) before applying f for a second time. Candidates who simplified at each step fared better than those who constructed complicated expressions before attempting to deal with them. Attention to detail was needed too and some candidates made things difficult for themselves by not adding or subtracting 3 at appropriate stages. After applying ff successfully, candidates were faced with the equation 2 + ln(N + 5) = ln(53e^2) and many were unable to deal with this correctly. All too common was a next step of e^2 + (N + 5) = 53e^2. A few candidates demonstrated a sound understanding of functions by taking the result in part (ii) to use e^N – 3 = f^{-1}f^{-1}[ln(53e^2)] as the method for finding the value of N.
		iii	Obtain 48 from correct work	A1	
		iii	Or 2:		
		iii	Apply f^{-1} to obtain e$^{\ln(53e2)-2}$ – 3	B1	
		iii	Attempt simplification of expression involving ln and e	M1	
		iii	Obtain f(e^N – 3) = 50	A1	
		iii	Apply f, or apply f^{-1} to right-hand side	M1	
		iii	Obtain 48	A1	
			Total	11	
3		i	State range of f is f(x) ≥ 3a or y ≥ 3a	B1	Allow f ≥ 3a or equiv expression in words but 3a to be included
		i	State range of g is all real numbers or equiv such as y ∈ R (real numbers)	B1	
		ii	State function is not 1 – 1 or different x-values give same y-value or equiv	B1	no credit for 'no inverse due to modulus' nor for 'cannot be reflected across y = x'
		ii	Obtain form k(y + 4a) or k(x + 4a)	M1	for non-zero constant k
		ii	Obtain $\frac{1}{5}(x+4a)$ or $\frac{1}{5}x+\frac{4}{5}a$	A1	Must finally be in terms of x
		iii	Either Attempt composition of functions the right way round	M1	Earned for 5(what they think f(x) is) – 4a
		iii	Obtain 5 |2x + a| + 11a = 31a or equiv	A1	

		iii	Or Apply their g^{-1} to $31a$	M1	
		iii	Obtain $\|2x + a\| + 3a = 7a$ or equiv	A1	
		iii	Either Solve $2x + a = 4a$ and obtain $\frac{3}{2}a$	B1 FT	Following their $\|2x + a\| = ka$
		iii	Solve linear equation in which signs of (their) $2x$ and (their) $4a$ are different	M1	Condone other sign slips
		iii	Obtain $-\frac{5}{2}a$	A1	And no others; obtaining $-\frac{5}{2}a$ and then concluding $\frac{5}{2}a$ is A0
		iii	Or Square both sides and obtain $4x^2 + 4ax - 15a^2 = 0$	B1 FT	Following their $\|2x + a\| = ka$
		iii	Solve 3-term quadratic equation to obtain two values	M1	Allow M1 if factorisation wrong but expansion gives correct first and third terms; allow M1 if incorrect use of formula involves only one error
		iii	Obtain $-\frac{5}{2}a, \quad \frac{3}{2}a$	A1	And no others; continuing from two correct answers to conclude $\frac{5}{2}a, \quad \frac{3}{2}a$ is A0 **Examiner's Comments** Identifying the range of f was not done well and $f(x) \geq 4a$ was a common wrong response. Candidates generally had more idea with g although some found it difficult to express their answer clearly. Provided the answer conveyed the idea of all real numbers, the mark was earned. The answer $-\infty \leq g(x) \leq \infty$ was frequently offered and accepted. But answers clearly referring to values of x were not accepted. The vast majority of candidates earned 2 marks in part (ii) for finding the inverse of g; the only errors to occur with any frequency were sign slips. The mark for explaining why f has no inverse was not earned so easily. A statement that f is not 1 – 1 or that f is many – one was sufficient to earn the mark. But, in some cases, there was confusion between many – one and one – many. Some responses were contradictory: f is a one – many function, i.e. one value of y gives many values of x. There were also many comments saying that f has no inverse because of the modulus or that f has no inverse because it cannot be reflected in the line $y = x$. There was good work seen in response to part (iii) with half of the candidates earning all the marks. The composition of the two functions was almost always carried out the right way round. Dealing with $\|2x + a\|$ presented some problems. A few candidates immediately replaced it with $(2x + a)^2$; others treated the modulus signs as brackets and proceeded to solve $5(2x + 4a) - 4a = 31a$, an approach which does give one of the solutions of the equation. For those candidates reaching the stage $\|2x + a\| = 4a$, it was encouraging to note that the majority proceeded without fuss to solve the two linear equations $2x + a = 4a$ and $2x + a = -4a$. Those opting to square both sides of $\|2x + a\| = 4a$ did not fare so well, not always being able to cope with the quadratic equation involving both x and a. A few candidates mistakenly decided to reject the answer $-\frac{5}{2}a$ at the end, apparently believing

					that the presence of modulus signs in the question meant that nothing could be negative.
			Total	10	
4		a	(0, 4]	B1(AO2.5) [1]	Do not allow $0 < f(x) \le 4$
		b	$f^{-1}(x) = \frac{8}{x} - 2$	M1(AO1.1a) A1(AO1.1) [2]	Obtain $\frac{8}{x} \pm 2$ — Allow in terms of y Obtain correct inverse function — Must now be in terms of x
		c	$x = \frac{8}{x+2}$ $x^2 + 2x - 8 = 0$ $x = 2$	M1(AO1.1a) A1(AO2.3) [2]	Equate two of x, $f(x)$ and $f^{-1}(x)$ and attempt to solve Obtain $x = 2$ only — A0 if $x = -4$ also given
			Total	5	
5			Reflection, stretch and translation (reflection) in the line $y = x$ (stretch) scale factor $\frac{1}{3}$ parallel to the x-axis (translation) $\begin{pmatrix} 0 \\ -5 \end{pmatrix}$	B1(AO2.5) B1(AO1.1) B1(AO1.1) B1(AO1.1) [4]	All three correct — Do not accept any other wording Accept 'in the x-direction' accept 'factor' or 'SF' for 'scale factor' — Do not accept 'in/on/across/up the x-axis' or '$\frac{1}{3}$ units' Accept '5 units in the negative y-direction' or '-5 units parallel to the y-axis' — Do not accept 'in/on/across/up the y-axis' Order of

					transformations must be correct for all 4 marks to be awarded
		Total	4		
6	a	$f(x) = c + 16 - (x - 4)^2$ $c + 16 = 19$ $c = 3$	M1* (AO 3.1a) M1d* (AO 1.1a) A1 (AO 1.1) [3]	Attempt to identify maximum point Link maximum point to 19 Solve to obtain $c = 3$	Full attempt to complete the square Could differentiate, equate to 0 and solve to get $8 - 2x = 0$, so $x = 4$ Link the constant term of their completed square to 19 – must involve c Allow equation or inequality (including incorrect inequality) If using differentiation then link f(their $x = 4$) to 19 A0 if given as inequality unless subsequently corrected Must come from fully correct working, so $f(x) = c + 16 - (x + 4)^2$, leading to $c + 16 = 19$ hence $c = 3$ is M1 M1 A0 **OR** M1* Attempt to use $b^2 - 4ac = 0$ on their attempt at $f(x) - 19 = 0$ M1d* Attempt to

					solve their $64 - 4(-1)(c - 19) = 0$ A1 Obtain $c = 3$ Examiner's Comments Many fully correct solutions to this question were seen, with candidates employing a variety of different methods. The most common approaches were to write f(x) in completed square form and equate the maximum value to 19, or to use differentiation to identify the maximum point. Some candidates attempted rearranged to obtain $f(x) - 19 \le 0$ and attempted to use the discriminant, but only the most able identified that the condition for repeated roots should then be used.
		b	$f(2) = c + 12$ $f(c + 12) = c + 8(c + 12) - (c + 12)^2$ $-48 - 15c - c^2 = 8$ $c^2 + 15c + 56 = 0$ $c = -7$, $c = -8$	B1 (AO 1.1) M1* (AO 1.2) M1d* (AO 1.1a) A1 (AO 2.1) [4]	Correct f(2) — Stated or implied by being used in later method Attempt correct composition of ff — Must be attempt at composition of functions so M0 for $\{f(2)\}^2$ Equate to 8 and rearrange to useable form — Expand and rearrange to a three term quadratic. Could be implied by the two correct roots Both correct values for c — BC **OR** for the first two marks **M1*** Attempt ff(x) ie attempt at $ff(x) = c + 8(c + 8x - x^2) - (c + 8x - x^2)^2$ **M1d*** Attempt ff(2) using their ff(x), which may no longer be correct Examiner's Comments

					All candidates understood the meaning of ff(2) and were able to attempt the correct process for the composition of functions. The more successful method was to first find f(2), simplify this to $c + 12$ and then attempt f($c + 12$). Candidates who attempted to find ff(x) before substituting $x = 2$ were more likely to make mistakes when simplifying their algebraic expression. It was expected that candidates would use their calculators to solve the quadratic equation, but the vast majority instead showed full detail of the method used.
			Total	7	
7		a	(i) $fg(x) = f(x^2 + 2) = (x^2 + 2)^3$	B1(AO 1.1)E [1]	**Examiner's Comments** Nearly all candidates correctly found the composite function fg(x) as $(x^2 + 2)^3$ although a number did expand the bracket in this part. Candidates are reminded that the number of marks available for a question or part–question are the best indicators to the amount of working and detail that is required.
			(ii) $gf(x) = g(x^3) = (x^3)^2 + 2 (= x^6 + 2)$	B1(AO 1.1)E [1]	No simplification required **Examiner's Comments** Once again this was nearly always done correctly with the most common errors being those minority of candidates who stated that $(x^2)^3$ was equal to either x^5 or x^8.
		b	DR $(x^2 + 2)^3 = (x^2)^3 + 3(x^2)^2 (2) + 3(x^2) (2)^2 + 2^3$	M1(AO 1.1)E	Binomial expansion of their $(x^2 + 2)^3$ – correct powers and coefficients; Allow one slip
			$fg(x) = x^6 + 6x^4 + 12x^2 + 8$	A1(AO 1.1)C	
			$fg(x) - gf(x) = 24 \Rightarrow 6x^4 + 12x^2 - 18 = 0$	A1(AO 2.1)C	
			$x^4 + 2x^2 - 3 = 0 \Rightarrow (x^2 - 1)(x^2 + 3) = 0$	M1(AO 1.1)C	Correct method for solving their; If M0 next two

			$x^2 + 3 = 0$ has no real solutions $x^2 - 1 = 0 \Rightarrow x = \pm 1$	A1(AO 2.4)A A1(AO 2.2a)A [6]	<table><tr><td>quadratic in x^2 $x^2 + 3 \neq 0$ is acceptable for this mark</td><td>marks become B marks</td></tr></table> <u>Examiner's Comments</u> There were a significant number of candidates that did not employ correct bracketing and mistakenly wrote $(x^2 + 2)^3 - x^6 + 2 =$ rather than $(x^2 + 2)^3 - (x^6 + 2) = 24 \Rightarrow (x^2 + 2)^3 - x^6 - 2 = 24$. The majority of candidates went for expanding $(x^2 + 2)^3$ by writing out the bracket three times rather than using the binomial expansion of $(a + b)^n$. While the majority rearranged their quartic into the form $6(x^2)^2 + 12x^2 + 12x^2 - 18 = 0$ there were a number of candidates did not show sufficient working in solving this quartic, even though the full question required detailed reasoning. Finally, it is expected at this level that as part of the detailed reasoning candidates should justify why $x^2 + 3 = 0$ did not provide any solutions.
			Total	8	