

1.

$$\frac{1 + 11x - 6x^2}{(x - 3)(1 - 2x)} \equiv A + \frac{B}{(x - 3)} + \frac{C}{(1 - 2x)}$$

(a) Find the values of the constants A , B and C .

(4)

$$f(x) = \frac{1 + 11x - 6x^2}{(x - 3)(1 - 2x)} \quad x > 3$$

(b) Prove that $f(x)$ is a decreasing function. $\rightarrow f'(x) < 0$

(3)

Question continued

a)

$$\frac{a}{b} + \frac{c}{d} + \frac{e}{f} = \frac{adf + cbf + ebf}{bdf}$$

$$\frac{1+11x-6x^2}{(x-3)(1-2x)} = \frac{A(x-3)(1-2x) + B(1-2x) + C(x-3)}{(x-3)(1-2x)}$$

$$\frac{a}{b} = \frac{c}{b} \Rightarrow a = c$$

$$1+11x-6x^2 = A(x-3)(1-2x) + B(1-2x) + C(x-3) \quad \checkmark$$

To find B, $x-3=0 \Rightarrow x=3$

$$1+11(3)-6(3)^2 = 0+B(-5)$$

$$-20 = -5B \Rightarrow B=4 \quad \checkmark$$

To find C, $1-2x=0 \Rightarrow 2x=1 \therefore x=0.5$

$$1+11(0.5)-6(0.5)^2 = 0+0-2.5C$$

$$5 = -2.5C$$

$$C = -2$$

$$1+11x-6x^2 = A(x-3)(1-2x) + 4(1-2x) - 2(x-3)$$

$$x=0, \quad 1+11(0)-6(0) = -3A + 4 + 6$$

$$1 = -3A + 10$$

$$-3A = -9$$

$$A = 3 \quad \checkmark$$

$$\frac{1+11x-6x^2}{(x-3)(1-2x)} = 3 + \frac{4}{(x-3)} - \frac{2}{(1-2x)} \quad \checkmark$$

Question continued

b)

$$f(x) = 3 + \frac{4}{(x-3)} - \frac{2}{(1-2x)}$$

if $y = A(f(x))^n$
 $\frac{dy}{dx} = Anx(f'(x))^{n-1}$

$$f(x) = 3 + 4(x-3)^{-1} - 2(1-2x)^{-1}$$

$$f'(x) = 0 - 4(x-3)^{-2} - 4(1-2x)^{-2} \quad \checkmark \checkmark$$

$$a^{-n} = \frac{1}{a^n}$$

$$= \frac{-4}{(x-3)^2} + \frac{-4}{(1-2x)^2}$$

$$\begin{aligned} (x-3)^2 &> 0 \quad \text{for all } x > 3 \\ (1-2x)^2 &> 0 \quad \text{for all } x > 3 \end{aligned}$$

$$\therefore -\frac{4}{(x-3)^2} < 0 \quad \text{and} \quad -\frac{4}{(1-2x)^2} < 0$$

we have shown that $-\frac{4}{(x-3)^2}$ and $-\frac{4}{(1-2x)^2} < 0$

for all $x > 3$. $f'(x)$ is the sum of 2

negative fractions. $\therefore f'(x) < 0 \Rightarrow f(x)$ is

decreasing. $\checkmark$

2.

$$f(x) = \frac{50x^2 + 38x + 9}{(5x + 2)^2(1 - 2x)} \quad x \neq -\frac{2}{5} \quad x \neq \frac{1}{2}$$

Given that $f(x)$ can be expressed in the form

$$\frac{A}{5x + 2} + \frac{B}{(5x + 2)^2} + \frac{C}{1 - 2x}$$

where A , B and C are constants

(a) (i) find the value of B and the value of C

(ii) show that $A = 0$

(4)

(b) (i) Use binomial expansions to show that, in ascending powers of x

$$f(x) = p + qx + rx^2 + \dots$$

where p , q and r are simplified fractions to be found.

(ii) Find the range of values of x for which this expansion is valid.

(7)

$$(a) \quad \frac{50x^2 + 38x + 9}{(5x + 2)^2(1 - 2x)} = \frac{A}{5x + 2} + \frac{B}{(5x + 2)^2} + \frac{C}{1 - 2x}$$

$$50x^2 + 38x + 9 = A(5x + 2)(1 - 2x) + B(1 - 2x) + C(5x + 2)^2 \quad (1)$$

$$x = \frac{1}{2} : 50\left(\frac{1}{2}\right)^2 + 38\left(\frac{1}{2}\right) + 9 = C(5\left(\frac{1}{2}\right) + 2)^2$$

$$\begin{aligned} 8\frac{1}{2} &= 8\frac{1}{4}C \\ C &= 2 \quad (1) \end{aligned}$$

$$\begin{aligned} x = -\frac{2}{5} : 50\left(-\frac{2}{5}\right)^2 + 38\left(-\frac{2}{5}\right) + 9 &= B(1 - 2\left(-\frac{2}{5}\right)) \\ \frac{9}{5} &= \frac{9}{5}B \\ B &= 1 \end{aligned}$$

$$\begin{aligned} x = 0 : 9 &= A(2)(1) + B(1) + C(2)^2 \\ 9 &= 2A + B + 4C \\ &= 2A + 1 + 4(2) \quad (1) \\ &= 2A + 9 \\ A &= 0 \quad (1) \end{aligned}$$

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Question continued

$$\frac{50x^2 + 38x + 9}{(5x+2)^2(1-2x)} \equiv \frac{1}{(5x+2)^2} + \frac{2}{(1-2x)} \quad *$$

$$(b)(i) \frac{1}{(5x+2)^2} \equiv (5x+2)^{-2} \equiv \left[2 \left(1 + \frac{5x}{2} \right) \right]^{-2} \equiv 2^{-2} \left(1 + \frac{5x}{2} \right)^{-2} \quad (1)$$

$$\approx 2^{-2} \left\{ 1 + \frac{(-2) \left(\frac{5x}{2} \right)}{1!} + \frac{(-2)(-3) \left(\frac{5x}{2} \right)^2}{2!} + \dots \right\} \quad (1)$$

$$= \frac{1}{4} \left\{ 1 - 5x + \frac{75x^2}{4} + \dots \right\}$$

$$= \frac{1}{4} - \frac{5x}{4} + \frac{75x^2}{16} + \dots \quad (1) \quad |x| < \frac{2}{5}$$

$$\frac{2}{1-2x} \equiv 2(1-2x)^{-1}$$

$$\approx 2 \left\{ 1 + \frac{(-1)(-2x)}{1!} + \frac{(-1)(-2)(-2x)^2}{2!} + \dots \right\} \quad (1)$$

$$= 2 \{ 1 + 2x + 4x^2 \}$$

$$= 2 + 4x + 8x^2 + \dots$$

$$\therefore f(x) = \frac{1}{4} - \frac{5x}{4} + \frac{75x^2}{16} + \dots + 2 + 4x + 8x^2 + \dots \quad (1)$$

$$= \frac{9}{4} + \frac{11x}{4} + \frac{203x^2}{16} + \dots \quad (1) \quad |x| < \frac{1}{2}$$

$$(ii) |x| < \frac{2}{5} \quad *$$

take the smaller value

$$\frac{2}{5} < \frac{1}{2}$$

(1)

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3. (a) Express $\frac{1}{P(11-2P)}$ in partial fractions.

(3)

A population of meerkats is being studied.

The population is modelled by the differential equation

$$\frac{dP}{dt} = \frac{1}{22} P(11-2P), \quad t \geq 0, \quad 0 < P < 5.5$$

where P , in thousands, is the population of meerkats and t is the time measured in years since the study began.

Given that there were 1000 meerkats in the population when the study began, $t=0$, $P=1$

- (b) determine the time taken, in years, for this population of meerkats to double,

(6)

- (c) show that

$$P = \frac{A}{B + Ce^{-\frac{1}{2}t}}$$

where A , B and C are integers to be found.

(3)

$$a) \frac{1}{P(11-2P)} = \frac{A}{P} + \frac{B}{11-2P} \quad - (1)$$

$$1 = A(11-2P) + BP$$

$$\begin{array}{ll} \text{When } P=0; & \text{When } P=\frac{11}{2}; \\ 1 = 11A & 1 = \frac{11}{2}B \\ A = \frac{1}{11} & B = \frac{2}{11} \end{array} \quad - (1)$$

$$= \frac{\frac{1}{11}}{P} + \frac{\frac{2}{11}}{11-2P} \quad (1) \quad \text{or} \quad \frac{1}{11P} + \frac{2}{11(11-2P)}$$

$$b) \frac{dP}{dt} = \frac{1}{22} P(11-2P)$$

$$\int \frac{22}{P(11-2P)} dP = \int dt \quad - (1)$$

$$\int \left(\frac{2}{P} + \frac{4}{11-2P} \right) dP = \int dt \quad - (1)$$

$$2 \ln P - 2 \ln(11-2P) = t + C \quad - (1)$$

$$\frac{1}{P(11-2P)} = \frac{\frac{1}{11}}{P} + \frac{\frac{2}{11}}{11-2P}$$

$$\frac{22}{P(11-2P)} = \frac{2}{P} + \frac{4}{11-2P}$$

$$\int \frac{1}{x} dx = \ln|x|$$

$$\frac{4 \ln(11-2P)}{-2} = -2 \ln(11-2P)$$

b) When $t=0$, $P=1$;

$$2\ln 1 - 2\ln 9 = C$$

$$C = -2\ln 9 \quad - (1)$$

$$2\ln \frac{1}{9} = 2\ln 9^{-1} \\ = -2\ln 9$$

$$2\ln P - 2\ln (11-2P) = t - 2\ln 9$$

$$\text{At } P=2; \quad - (1)$$

$$2\ln 2 - 2\ln 7 = t - 2\ln 9$$

$$t = 1.89 \text{ years (3 s.f.)} \quad - (1)$$

c) $2\ln P - 2\ln (11-2P) = t - 2\ln 9$

$$2\ln \left(\frac{P}{11-2P} \right) = t - 2\ln 9$$

$$2\ln \left(\frac{9P}{11-2P} \right) = t$$

$$\ln \left(\frac{9P}{11-2P} \right) = \frac{1}{2}t \quad - (1)$$

$$\frac{9P}{11-2P} = e^{\frac{1}{2}t}$$

$$9P = 11e^{\frac{1}{2}t} - 2Pe^{\frac{1}{2}t} \quad - (1)$$

$$P(9 + 2e^{\frac{1}{2}t}) = 11e^{\frac{1}{2}t}$$

$$P = \frac{11e^{\frac{1}{2}t}}{9 + 2e^{\frac{1}{2}t}} \times \frac{e^{-\frac{1}{2}t}}{e^{-\frac{1}{2}t}}$$

$$P = \frac{11}{9e^{-\frac{1}{2}t} + 2}$$

$$P = \frac{11}{2 + 9e^{-\frac{1}{2}t}}$$

$$A = 11$$

$$B = 2 \quad - (1)$$

$$C = 9$$

4. The curve C with equation

$$y = \frac{p - 3x}{(2x - q)(x + 3)} \quad x \in \mathbb{R}, x \neq -3, x \neq 2$$

where p and q are constants, passes through the point $\left(3, \frac{1}{2}\right)$ and has two vertical asymptotes with equations $x = 2$ and $x = -3$

(a) (i) Explain why you can deduce that $q = 4$

(ii) Show that $p = 15$

(3)

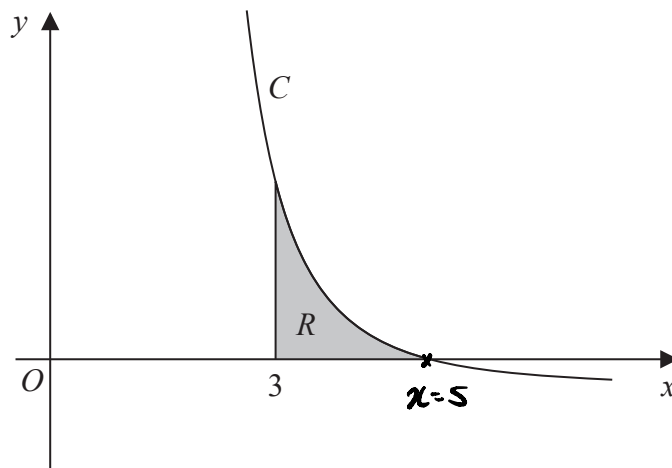


Figure 4

Figure 4 shows a sketch of part of the curve C . The region R , shown shaded in Figure 4, is bounded by the curve C , the x -axis and the line with equation $x = 3$

(b) Show that the exact value of the area of R is $a \ln 2 + b \ln 3$, where a and b are rational constants to be found.

(8)

a i) Vertical asymptotes when $(2x - q)(x + 3) = 0$

$$\begin{array}{lcl} \downarrow & & \downarrow \\ 2x - q = 0 & & x + 3 = 0 \\ 2x = q & & x = -3 \end{array}$$

① $x = 2$

$$2(2) = q$$

$$\therefore q = 4 \text{ as needed}$$

a ii) $y = \frac{p - 3x}{(2x - 4)(x + 3)}$

$$\frac{1}{2} = \frac{p - 3(3)}{(2(3) - 4)(3 + 3)} = \frac{p - 9}{12}$$

$2 \times 6 = 12$ ① $(3, \frac{1}{2})$

$$\frac{1}{2} = \frac{p - 9}{12}$$

$$6 = p - 9 \quad \text{①}$$

$$p = 6 + 9$$

$$\therefore p = 15 \text{ as needed}$$

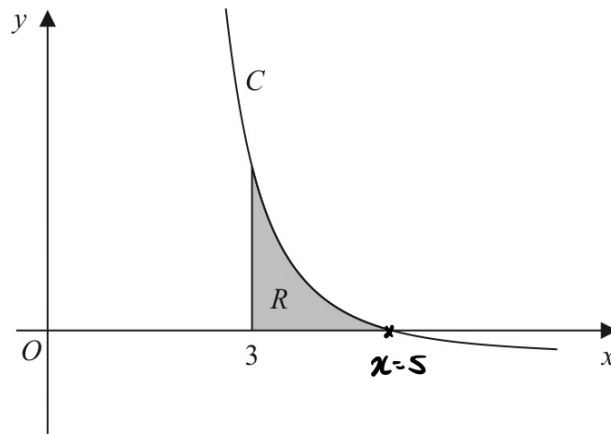


Figure 4

Figure 4 shows a sketch of part of the curve C . The region R , shown shaded in Figure 4, is bounded by the curve C , the x -axis and the line with equation $x = 3$

- (b) Show that the exact value of the area of R is $a \ln 2 + b \ln 3$, where a and b are rational constants to be found.

(8)

$$\frac{15-3x}{(2x-4)(x+3)} = \frac{1.8}{(2x-4)} - \frac{2.4}{(x+3)} \quad \textcircled{1} \quad \int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$$

$$\int_3^5 \frac{1.8}{(2x-4)} - \frac{2.4}{(x+3)} dx = \left[0.9 \ln|2x-4| - 2.4 \ln|x+3| \right]_3^5$$

$$f(x) = 2x-4$$

$$f'(x) = 2$$

$$\frac{1.8}{2} = 0.9$$

$$f(x) = x+3$$

$$f'(x) = 1$$

$$\frac{2.4}{1} = 2.4$$

④

$$= 0.9 \ln|2(5)-4| - 2.4 \ln|5+3| - \left[0.9 \ln|2(3)-4| - 2.4 \ln|3+3| \right]$$

$$= 0.9 \ln|6| - 2.4 \ln|8| - 0.9 \ln|2| + 2.4 \ln|6|$$

$$= 3.3 \ln|6| - 2.4 \ln|8| - 0.9 \ln|2|$$

$$= 3.3(\ln|3| + \ln|2|) - 2.4 \ln|2^3| - 0.9 \ln|2|$$

$$= 3.3 \ln|3| + 3.3 \ln|2| - 7.2 \ln|2| - 0.9 \ln|2|$$

$$= 3.3 \ln|3| - 4.8 \ln|2| \quad \textcircled{1}$$

using log laws
 $\ln(a \times b) = \ln a + \ln b$
 $\ln a^x = x \ln a$



5. (a) Use the substitution $x = u^2 + 1$ to show that

$$\int_5^{10} \frac{3 \, dx}{(x-1)(3+2\sqrt{x-1})} = \int_p^q \frac{6 \, du}{u(3+2u)}$$

where p and q are positive constants to be found.

(4)

a) $\int_5^{10} \frac{3}{(\underbrace{x-1}_{u^2+1})(3+2\underbrace{\sqrt{x-1}}_u)} \, dx$ Integration by Substitution:

$\hookrightarrow = \int_2^3 \frac{3}{(u^2+1-1)(3+2u)} \cdot 2u \, du$ ①

$x = u^2 + 1$
 $u^2 = x - 1 \Rightarrow u = \sqrt{x-1}$
 $du = \frac{1}{2\sqrt{x-1}} \, dx$

$= \int_2^3 \frac{6u}{u^2 \cdot (3+2u)} \, du$ New Limits: For $x=5$, $u = \sqrt{5-1} = \sqrt{4} = 2$
 $x=10$, $u = \sqrt{10-1} = \sqrt{9} = 3$ } new limits ①

$= \int_2^3 \frac{6}{u(3+2u)} \, du$ as required, with $p=2$ and $q=3$. ①

(b) Hence, using algebraic integration, show that

$$\int_5^{10} \frac{3 \, dx}{(x-1)(3+2\sqrt{x-1})} = \ln a$$

where a is a rational constant to be found.

(6)

b) From part a : $\int_5^{10} \frac{3}{(x-1)(3+2\sqrt{x-1})} \, dx = \int_2^3 \frac{6}{u(3+2u)} \, du$

Partial Fractions : $\frac{6}{u(3+2u)} = \frac{A}{u} + \frac{B}{3+2u}$

$$\Rightarrow 6 = A(3+2u) + Bu$$

let $u=0 \Rightarrow 6 = 3A \Rightarrow A = \underline{2}$ and let $u=1 \Rightarrow 6 = 10 + B$

$$\Rightarrow \int_2^3 \frac{6}{u(3+2u)} = \int_2^3 \frac{2}{u} - \frac{4}{3+2u} \, du = [2\ln(u) - 2\ln(3+2u)]_2^3 \quad \text{②} \quad \Rightarrow B = \underline{-4} \quad \text{①}$$

$$= [2\ln(3) - 2\ln(9)] - [2\ln(2) - 2\ln(7)] - 4 \int \frac{1}{3+2u} \, du = \frac{-4 \cdot \ln(3+2u)}{2} = -2\ln(3+2u)$$

$$\ln(a^b) = b\ln(a)$$

$$2\ln(9) = 2\ln(3^2) = 4\ln(3)$$

$$\log(a+b) = \log(ab)$$

$$\log(a-b) = \log\left(\frac{a}{b}\right)$$

$$= -2\ln(3) - 2\ln(2) + 2\ln(7)$$

$$= 2\ln\left(\frac{7}{3}\right) - 2\ln(2)$$

$$= 2\ln\left(\frac{7}{3 \times 2}\right) = 2\ln\left(\frac{7}{6}\right) = \ln\left(\frac{7^2}{6^2}\right) = \ln\left(\frac{49}{36}\right) = \ln(a) \quad \text{①}$$

where $a = \underline{\underline{\frac{49}{36}}}$