

1. Given

$$f(x) = e^x, \quad x \in \mathbb{R}$$

$$g(x) = 3 \ln x, \quad x > 0, x \in \mathbb{R}$$

(a) find an expression for $gf(x)$, simplifying your answer.

(2)

(b) Show that there is only one real value of x for which $gf(x) = fg(x)$

(3)

$$a) gf(x) = g(f(x)) = g(e^x)$$

$$\ln x^n = n \ln x$$

$$\ln e = 1$$

$$g(e^x) = 3 \ln e^x \quad - (1)$$

$$= 3x \ln e$$

$$= 3x \times 1$$

$$= 3x$$

$$gf(x) = 3x \quad - (1)$$

$$b) fg(x) = f(g(x)) = f(3 \ln x)$$

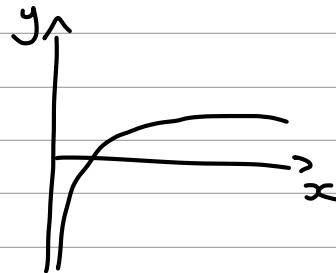
$$e^{\ln x} = x$$

$$\begin{aligned} f(3 \ln x) &= e^{3 \ln x} \\ &= e^{\ln x^3} \\ &= x^3 \end{aligned}$$

$$3x = x^3 \quad - (1)$$

$$x^2 = 3$$

$$x = \pm \sqrt{3} \quad - (1)$$



$y = \ln x$ isn't defined when $x \leq 0$

If $y = \ln x$ isn't defined at $x \leq 0$, $x = \sqrt{3}$ and so there's one real value. $- (1)$

2.

$$g(x) = \frac{2x+5}{x-3} \quad x \geq 5$$

(a) Find $gg(5)$.

(2)

(b) State the range of g .

(1)

(c) Find $g^{-1}(x)$, stating its domain.

(3)

$$a) \quad gg(5) = \frac{40}{9}$$

$$b) \quad 2 < g(x) \leq \frac{15}{2}$$

$$c) \quad g(x) = \frac{2x+5}{x-3}$$

$$y = \frac{2x+5}{x-3}$$

$$y(x-3) = 2x+5$$

$$xy - 3y = 2x+5$$

$$xy - 2x = 3y + 5$$

$$x(y-2) = 3y+5$$

$$x = \frac{3y+5}{y-2}$$

$$g^{-1}(x) = \frac{3x+5}{x-2}$$

$$2 < x \leq \frac{15}{2}$$

3.

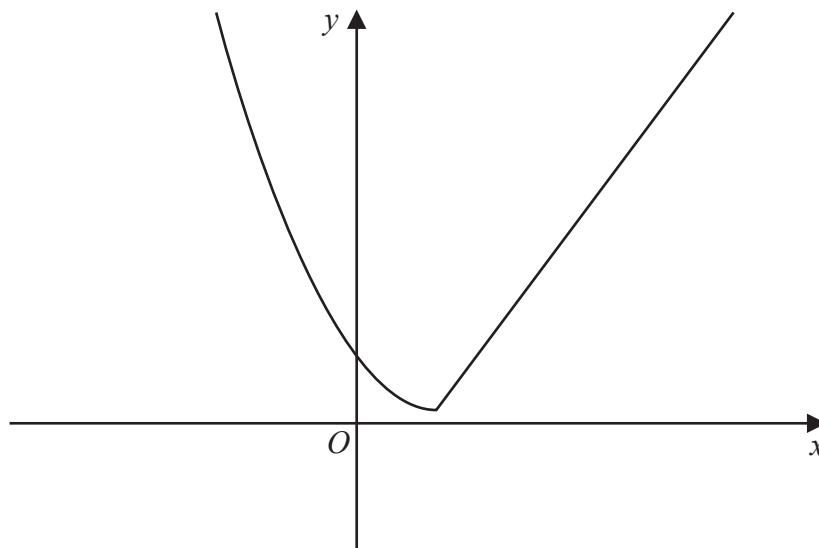
**Figure 4**

Figure 4 shows a sketch of the graph of $y = g(x)$, where

$$g(x) = \begin{cases} (x-2)^2 + 1 & x \leq 2 \\ 4x - 7 & x > 2 \end{cases}$$

(a) Find the value of $g(0)$.

(2)

(b) Find all values of x for which

$$g(x) > 28$$

(4)

The function h is defined by

$$h(x) = (x-2)^2 + 1 \quad x \leq 2$$

(c) Explain why h has an inverse but g does not.

(1)

(d) Solve the equation

$$h^{-1}(x) = -\frac{1}{2}$$

(3)

$$a) \quad g(x) = \begin{cases} (x-2)^2 + 1 & x \leq 2 \\ 4x - 7 & x > 2 \end{cases}$$

$$g(0) \Rightarrow x = 0 \leq 2 \Rightarrow g(x) = (x-2)^2 + 1$$

$$\Rightarrow g(0) = (0-2)^2 + 1 \Rightarrow g(0) = 5 \quad (1)$$

$$g(5) \Rightarrow x = 5 > 2 \Rightarrow g(x) = 4x - 7$$

$$\Rightarrow g(g(0)) = 4(g(0)) - 7$$

$$= 4(5) - 7 \Rightarrow \underline{g(g(0)) = 13} \quad (1)$$

$$b) \quad g(x) = \begin{cases} (x-2)^2 + 1 & x \leq 2 \\ 4x - 7 & x > 2 \end{cases}$$

$$4x - 7 = 28 \quad (1)$$

$$\text{and } (x-2)^2 + 1 = 28$$

$$\Rightarrow \frac{4 \pm \sqrt{(-4)^2 - 4(1)(-23)}}{2}$$

$$\Rightarrow 4x = 35$$

$$x^2 - 4x + 5 = 28$$

$$\Rightarrow x = \frac{35}{4} \quad (1)$$

$$x^2 - 4x - 23 = 0$$

$\rightarrow$ Quadratic formula

$$\Rightarrow x = \frac{4 \pm \sqrt{108}}{2} = 2 \pm 3\sqrt{3} \quad (1)$$

$$g\left(\frac{35}{4}\right) = 28, \text{ let's have } g\left(\frac{35}{4} + 0.1\right) = 28.4 > 28 \Rightarrow x > \frac{35}{4}$$

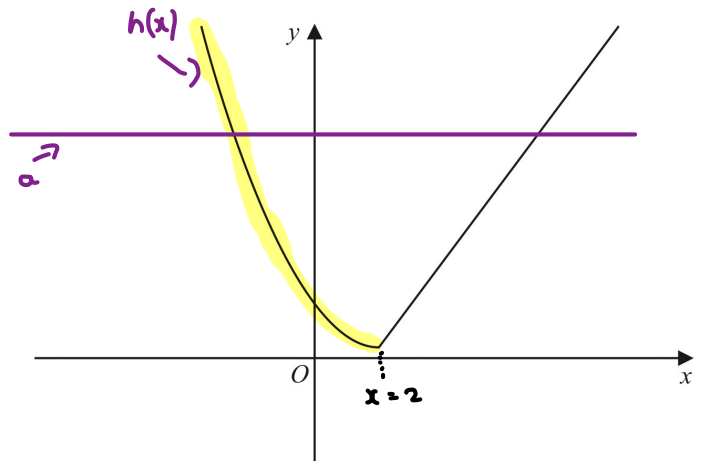
$$g(2 + 3\sqrt{3}) = 21.78 \neq 28 \Rightarrow x = 2 + 3\sqrt{3} \text{ is not a solution / critical value}$$

$$g(2 - 3\sqrt{3}) = 28, \text{ then } g(2 - 0.1 - 3\sqrt{3}) = 29.05 > 28 \Rightarrow x < \underline{2 - 3\sqrt{3}}$$

$$\Rightarrow g(x) > 28 \text{ for } x < 2 - 3\sqrt{3} \text{ and } x > \underline{\frac{35}{4}} \quad (1)$$

c) h is one-to-one, so it has an inverse by horizontal line test (the line only cuts $h(x)$ once hence one-to-one)

- g is many-to-one, so it does not have an inverse, since the line a intercepts the curve $g(x)$ at more than one point. ①



d)

$$h(x) = (x-2)^2 + 1 \quad x \geq 2$$

$$f^{-1}(x) = a \quad \text{then} \quad x = g(a)$$

$$h^{-1}(x) = -\frac{1}{2} \Rightarrow x = h\left(-\frac{1}{2}\right) \quad ①$$

$$\Rightarrow x = \left(-\frac{1}{2} - 2\right)^2 + 1 = \frac{29}{4} \Rightarrow x = \underline{\underline{7.25}} \quad ①$$

4. The function f is defined by

$$f(x) = \frac{3x-7}{x-2} \quad x \in \mathbb{R}, x \neq 2$$

(a) Find $f^{-1}(7)$

(2)

a) $f(x) = \frac{3x-7}{x-2} \Rightarrow y = \frac{3x-7}{x-2}$ ① Swap x and y
② Solve for y

$$\begin{aligned} \Rightarrow x &= \frac{3y-7}{y-2} \Rightarrow x(y-2) = 3y-7 \\ &\Rightarrow xy-2x = 3y-7 \\ &\Rightarrow xy-3y = 2x-7 \\ &\Rightarrow y(x-3) = 2x-7 \\ &\Rightarrow y = \frac{2x-7}{x-3} \end{aligned}$$

$$\Rightarrow f^{-1}(x) = \frac{2x-7}{x-3} \Rightarrow f^{-1}(7) = \frac{2(7)-7}{7-3} = \frac{7}{4} \quad \text{①}$$

$$\Rightarrow \underline{\underline{f^{-1}(7) = \frac{7}{4}}} \quad \text{①}$$

(b) Show that $ff(x) = \frac{ax+b}{x-3}$ where a and b are integers to be found.

(3)

b)

$$f(x) = \frac{3x-7}{x-2}$$

$$\Rightarrow ff(x) = \frac{3f(x)-7}{f(x)-2}$$

$$= \frac{2x-7}{x-2} \times \frac{x-2}{x-3} \quad \text{①}$$

$$3f(x) = 3\left(\frac{3x-7}{x-2}\right) = \frac{9x-21}{x-2}$$

$$3f(x)-7 = \frac{9x-21}{x-2} - \frac{7}{1} = \frac{9x-21-7x+14}{x-2}$$

$$= \frac{2x-7}{x-2} \quad (\text{numerator})$$

$$f(x)-2 = \frac{3x-7}{x-2} - \frac{2}{1} = \frac{3x-7-2x+4}{x-2}$$

$$\Rightarrow \underline{\underline{ff(x) = \frac{2x-7}{x-3} = \frac{ax+b}{x-3}}} \quad \text{as required with } a=2 \text{ and } b=-7. \quad \text{①}$$

5. The functions f and g are defined by

$$f(x) = 7 - 2x^2 \quad x \in \mathbb{R}$$

$$g(x) = \frac{3x}{5x-1} \quad x \in \mathbb{R} \quad x \neq \frac{1}{5}$$

(a) State the range of f

(1)

(b) Find $gf(1.8)$

(2)

(c) Find $g^{-1}(x)$

(2)

a) $y \leq 7$ ① when $x = 0$, $y = 7 - 2(0^2) = 7$
Then as x increases OR decreases, y will decrease
So y will always be less than or equal to 7.

b) $f(1.8) = 7 - 2(1.8^2)$ or find compound function $gf(x)$
 $= 0.52$
 $gf(1.8) = g(0.52)$ ①
 $g(0.52) = \frac{3 \times 0.52}{5 \times 0.52 - 1}$
 $= 0.975$ ①
 $\hookrightarrow gf(x) = \frac{3(7-2x^2)}{5(7-2x^2) - 1}$
 $= \frac{21-6x^2}{34-10x^2}$
 $\hookrightarrow$ then substitute $x = 1.8$
 $gf(1.8) = \frac{21-6(1.8^2)}{34-10(1.8^2)}$
 $= 0.975$

c) $y = \frac{3x}{5x-1}$ $\leftarrow$ rearrange to make x the subject
 $y(5x-1) = 3x$ $\div (5x-1)$
 $5xy - y = 3x$ $\div +y, -3x$
 $5xy - 3x = y$
 $x(5y-3) = y$ ① take x out as a factor
 $x = \frac{y}{5y-3}$ $\Rightarrow g^{-1}(x) = \frac{x}{5x-3}$ ①
swap x for $g^{-1}(x)$ and y for x

