Questions

Q1.

In this question you should show all stages of your working.

Solutions relying on calculator technology are not acceptable.

Using algebra, solve the inequality

$$x^2 - x > 20$$

writing your answer in set notation.

(Total for question = 3 marks)

Q2.

(a) Prove that for all positive values of *x* and *y*

$$\sqrt{xy} \leq \frac{x+y}{2}$$

(2)

(b) Prove by counter example that this is not true when *x* and *y* are both negative.

(1)

(Total for question = 3 marks)

Q3.

(i) Show that $x^2 - 8x + 17 > 0$ for all real values of x

(3)

(ii) "If I add 3 to a number and square the sum, the result is greater than the square of the original number."

State, giving a reason, if the above statement is always true, sometimes true or never true.

(2)

(Total for question = 5 marks)

Q4.

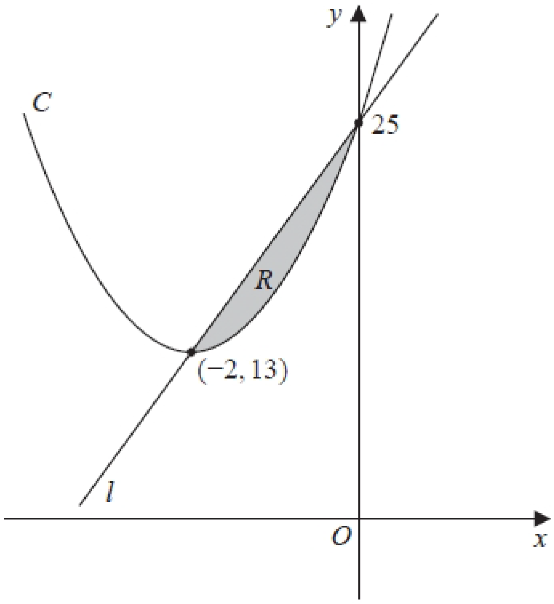

Figure 1

Figure 1 shows a sketch of a curve C with equation $y = f(x)$ and a straight line l.

The curve C meets l at the points (−2, 13) and (0, 25) as shown.

The shaded region R is bounded by C and l as shown in Figure 1.

Given that

- $f(x)$ is a quadratic function in x
- (−2, 13) is the minimum turning point of $y = f(x)$

use inequalities to define R.

(Total for question = 5 marks)

Mark Scheme

Q1.

Question	Scheme	Marks	AOs
	Finds critical values $x^2 - x > 20 \Rightarrow x^2 - x - 20 > 0 \Rightarrow x = (5, -4)$	M1	1.1b
	Chooses outside region for their values Eg. $x > 5, x < -4$	M1	1.1b
	Presents solution in set notation $\{x : x < -4\} \cup \{x : x > 5\}$ oe	A1	2.5
		(3)	
			(3 marks)

Notes

M1: Attempts to find the critical values using an algebraic method. Condone slips but an allowable method should be used and two critical values should be found

M1: Chooses the outside region for their critical values. This may appear in incorrect inequalities such as $5 < x < -4$

A1: Presents in set notation as required $\{x : x < -4\} \cup \{x : x > 5\}$ Accept $\{x < -4 \cup x > 5\}$.

Do not accept $\{x < -4,\ x > 5\}$

Note: If there is a contradiction of their solution on different lines of working do not penalise intermediate working and mark what appears to be their final answer.

Q2.

Question	Scheme	Marks	AOs
(a) Way 1	Since x and y are positive, their square roots are real and so $(\sqrt{x}-\sqrt{y})^2 \geq 0$ giving $x - 2\sqrt{x}\sqrt{y} + y \geq 0$	M1	2.1
	$\therefore\ 2\sqrt{xy} \leq x + y$ provided x and y are positive and so $\sqrt{xy} \leq \frac{x+y}{2}$ *	A1*	2.2a
		(2)	
Way 2 Longer method	Since $(x-y)^2 \geq 0$ for real values of x and y, $x^2 - 2xy + y^2 \geq 0$ and so $4xy \leq x^2 + 2xy + y^2$ i.e. $4xy \leq (x+y)^2$	M1	2.1
	$\therefore\ 2\sqrt{xy} \leq x + y$ provided x and y are positive and so $\sqrt{xy} \leq \frac{x+y}{2}$ *	A1*	2.2a
		(2)	
(b)	Let $x = -3$ and $y = -5$ then LHS $= \sqrt{15}$ and RHS$= -4$ so as $\sqrt{15} > -4$ result does not apply	B1	2.4
		(1)	
			(3 marks)

Notes

(a) M1 : Need two stages of the three stage argument involving the three stages, squaring, square rooting terms and rearranging.

A1*: Need all three stages making the correct deduction to achieve the printed result.

(b) B1 : Chooses two negative values and substitutes, then states conclusion

Q3.

Question	Scheme	Marks	AOs
(i)	$x^2 - 8x + 17 = (x-4)^2 - 16 + 17$	M1	3.1a
	$= (x-4)^2 + 1$ with comment (see notes)	A1	1.1b
	As $(x-4)^2 \geqslant 0 \Rightarrow (x-4)^2 + 1 \geqslant 1$ hence $x^2 - 8x + 17 > 0$ for all x	A1	2.4
		(3)	
(ii)	For an explanation that it may not always be true Tests say $x = -5$ $(-5+3)^2 = 4$ whereas $(-5)^2 = 25$	M1	2.3
	States sometimes true and gives reasons Eg. when $x = 5$ $(5+3)^2 = 64$ whereas $(5)^2 = 25$ True When $x = -5$ $(-5+3)^2 = 4$ whereas $(-5)^2 = 25$ Not true	A1	2.4
		(2)	
			(5 marks)

Notes

(i) Method One: Completing the Square

M1: For an attempt to complete the square. Accept $(x-4)^2$ ...

A1: For $(x-4)^2+1$ with either $(x-4)^2 \geqslant 0, (x-4)^2+1 \geqslant 1$ or min at (4,1). Accept the inequality statements in words. Condone $(x-4)^2>0$ or a squared number is always positive for this mark.

A1: A fully written out solution, with correct statements and no incorrect statements. There must be a valid reason and a conclusion

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$x^2-8x+17$ — scores M1 A1 A1

$=(x-4)^2+1 \geqslant 1$ as $(x-4)^2 \geqslant 0$

Hence $(x-4)^2+1>0$

$x^2-8x+17>0$ — scores M1 A1 A1

$(x-4)^2+1>0$

This is true because $(x-4)^2 \geqslant 0$ and when you add 1 it is going to be positive

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$x^2-8x+17>0$ — scores M1 A1 A0

$(x-4)^2+1>0$

which is true because a squared number is positive — incorrect and incomplete

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$x^2-8x+17=(x-4)^2+1$ — scores M1 A1 A0

Minimum is (4,1) so $x^2-8x+17>0$ — correct but not explained

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$x^2-8x+17=(x-4)^2+1$ — scores M1 A1 A1

Minimum is (4,1) so as $1>0 \Rightarrow x^2-8x+17>0$ — correct and explained

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$x^2-8x+17>0$ — scores M1 A0 (no explanation) A0

$(x-4)^2+1>0$

Method Two: Use of a discriminant

M1: Attempts to find the discriminant b^2-4ac with a correct a, b and c which may be within a quadratic formula. You may condone missing brackets.

A1: Correct value of $b^2-4ac=-4$ **and** states or shows curve is U shaped (or intercept is (0,17)) or equivalent such as +ve x^2 etc

A1: Explains that as $b^2-4ac<0$, there are no roots, and curve is U shaped then $x^2-8x+17>0$

Method Three: Differentiation

M1: Attempting to differentiate and finding the turning point. This would involve attempting to find $\frac{dy}{dx}$, then setting it equal to 0 and solving to find the x value and the y value.

A1: For differentiating $\frac{dy}{dx}=2x-8 \Rightarrow (4,1)$ is the **turning point**

A1: Shows that (4,1) is the minimum point (second derivative or U shaped), hence $x^2-8x+17>0$

Method 4: Sketch graph using calculator

M1: Attempting to sketch $y = x^2 - 8x + 17$, U shape with minimum in quadrant one

A1: As above with minimum at (4,1) marked

A1: Required to state that quadratics only have one turning point and as "1" is above the x-axis then $x^2 - 8x + 17 > 0$

(ii)

Numerical approach

Do not allow any marks if the student just mentions "positive" and "negative" numbers. Specific examples should be seen calculated if a numerical approach is chosen.

M1: Attempts a value (where it is not true) and shows/implies that it is not true for that value.

For example, for -4 : $(-4+3)^2 > (-4)^2$ and indicates not true (states not true, ×)

or writing $(-4+3)^2 < (-4)^2$ is sufficient to imply that it is not true

A1: Shows/implies that it can be true for a value **AND** states sometimes true.

For example for +4 : $(4+3)^2 > 4^2$ and indicates true ✓

or writing $(4+3)^2 > 4^2$ is sufficient to imply this is true following $(-4+3)^2 < (-4)^2$

condone incorrect statements following the above such as 'it is only true for positive numbers' as long as they state "sometimes true" and show both cases.

Algebraic approach

M1: Sets the problem up algebraically Eg. $(x+3)^2 > x^2 \Rightarrow x > k$ Any inequality is fine. You may condone one error for the method mark. Accept $(x+3)^2 > x^2 \Rightarrow 6x + 9 > 0$ oe

A1: States sometimes true and states/implies true for $x > -\frac{3}{2}$ or states/implies not true for $x \le -\frac{3}{2}$ In both cases you should expect to see the statement "sometimes true" to score the A1

Q4.

Question	Scheme	Marks	AOs
	Attempts equation of line Eg Substitutes $(-2,13)$ into $y = mx + 25$ and finds m	M1	1.1b
	Equation of l is $y = 6x + 25$	A1	1.1b
	Attempts equation of C Eg Attempts to use the intercept $(0,25)$ within the equation $y = a(x \pm 2)^2 + 13$, in order to find a	M1	3.1a
	Equation of C is $y = 3(x+2)^2 + 13$ or $y = 3x^2 + 12x + 25$	A1	1.1b
	Region R is defined by $3(x+2)^2 + 13 < y < 6x + 25$ o.e.	B1ft	2.5
		(5)	
			(5 marks)

The first two marks are awarded for finding the equation of the line

M1: Uses the information in an attempt to find an equation for the line l.

E.g. Attempt using two points: Finds $m = \pm\frac{25-13}{2}$ and uses of one of the points in their $y = mx + c$ or equivalent to find c. Alternatively uses the intercept as shown in main scheme.

A1: $y = 6x + 25$ seen or implied. This alone scores the first two marks. Do not accept $l = 6x + 25$

It must be in the form $y = ...$ but the correct equation can be implied from an inequality. E.g. $.... < y < 6x + 25$

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The next two marks are awarded for finding the equation of the curve

M1: A complete method to find the constant a in $y = a(x \pm 2)^2 + 13$ or the constants a, b in $y = ax^2 + bx + 25$.

An alternative to the main scheme is deducing equation is of the form $y = ax^2 + bx + 25$ and setting and solving a pair of simultaneous equations in a and b using the point $(-2, 13)$ the gradient being 0 at $x = -2$. Condone slips. Implied by $C = 3x^2 + 12x + 25$ or $3x^2 + 12x + 25$

FYI the correct equations are $13 = 4a - 2b + 25\ (2a - b = -6)$ and $-4a + b = 0$

A1: $y = 3(x+2)^2 + 13$ or equivalent such as $y = 3x^2 + 12x + 25$, $\mathrm{f}(x) = 3(x+2)^2 + 13$.

Do not accept $C = 3x^2 + 12x + 25$ or just $3x^2 + 12x + 25$ for the A1 but may be implied from an inequality or from an attempt at the area, E.g. $\int 3x^2 + 12x + 25\,\mathrm{d}x$

B1ft: Fully defines the region R. Follow through on their equations for l and C.

Allow strict or non -strict inequalities as long as they are used consistently.

E.g. Allow for example "$3(x+2)^2 + 13 < y < 6x + 25$" "$3(x+2)^2 + 13 \leqslant y \leqslant 6x + 25$"

Allow the inequalities to be given separately, e.g. $y < 6x + 25$, $y > 3(x+2)^2 + 13$. Set notation may be used so $\{(x, y) : y > 3(x+2)^2 + 13\} \cap \{(x, y) : y < 6x + 25\}$ is fine but condone with or without any of $(x, y) \leftrightarrow y \leftrightarrow x$

Incorrect examples include "$y < 6x + 25$ **or** $y > 3(x+2)^2 + 13$", $\{(x, y) : y > 3(x+2)^2 + 13\} \cup \{(x, y) : y < 6x + 25\}$

Values of x could be included but they must be correct. So $3(x+2)^2 + 13 < y < 6x + 25, x < 0$ is fine

If there are multiple solutions mark the final one.