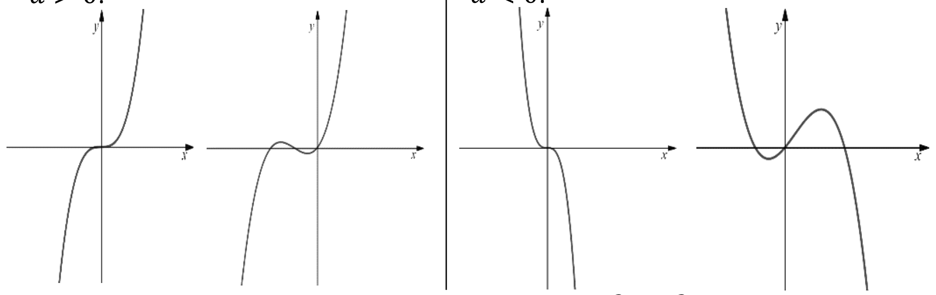
Graphs and Transformation Cheat Sheet

Cubic & Quartic Graphs

Cubic functions come in the form of $f(x) = ax^3 + bx^2 + cx + d$. Quartic functions come in the form of $f(x) = ax^4 + bx^3 + cx^2 + dx + e$, where a, b, c, d & e are all real numbers and where $a \neq 0$.

Similar to sketching quadratic graphs, cubic graphs can also be represented on a plot. There are two main basic shapes for cubic graphs if

$a > 0$: $a < 0$:

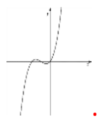


Example 1: Sketch the curve with the equation $y = x^3 + x^2 - 2x$.

Before we start sketching, we need to know

- the general shape of the cubic equation
- the location of the roots of the equation

We can compare the function to a general function of $f(x) = ax^3 + bx^2 + cx + d$ to determine the shape of the curve.

$a = 1$ and $1 > 0$ hence the shape of the graph would look like  .

The next step would be to find the roots of the functions so we can determine where it would cross the x axis. To do this we need to solve the quadratic equation of

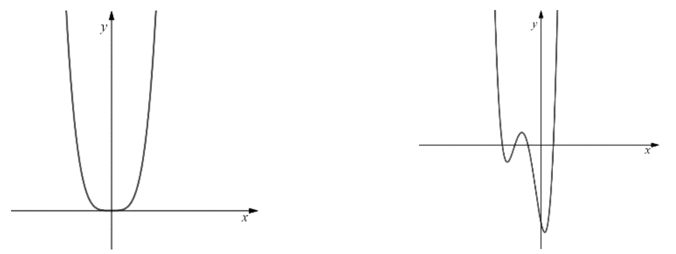
$$x^3 + x^2 - 2x = 0$$
$$x(x^2 + x - 2) = x(x+2)(x-1) = 0$$
$$x = 0 \text{ and } x - 1 = 0 \text{ and } x + 2 = 0$$

Hence the roots of the function are at:

$$x = 0 \text{ and } x = 1 \text{ and } x = -2$$

At coordinates $(0,0)$, $(1,0)$ & $(-2,0)$ are where the x intercepts will be located.

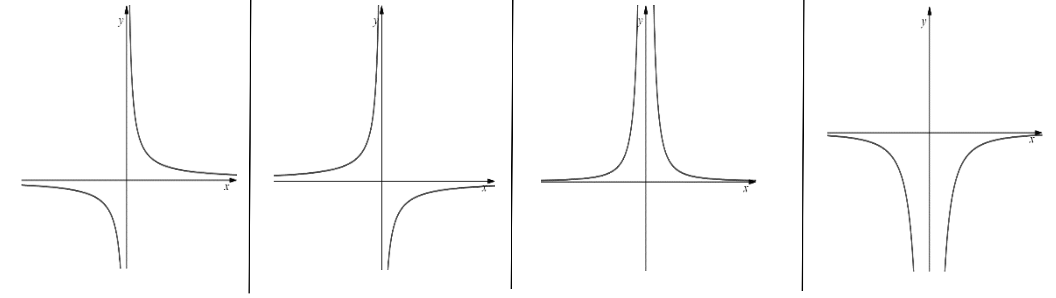
The same method can be applied to sketching graphs of quartic functions. The basic shapes of these graphs come in the form of:



Reciprocal graphs

Reciprocal graphs that come in the form of $f(x) = \frac{a}{x}$ or $f(x) = \frac{a}{x^2}$, where a is any real number, can also be sketched by considering their asymptotes. The graphs of $y = \frac{a}{x}$ and $y = \frac{a}{x^2}$ both have asymptotes at $x = 0$ and $y = 0$. The basic shapes of these reciprocal graphs can be illustrated as:

$y = \frac{a}{x}$, where $a > 0$ | $y = \frac{a}{x}$, where $a < 0$ | $y = \frac{a}{x^2}$, where $a > 0$ | $y = \frac{a}{x^2}$, where $a > 0$



Points of Intersection

Multiple functions can be sketched on a single graph to show the points of intersection, which represent the solutions to respective equations.

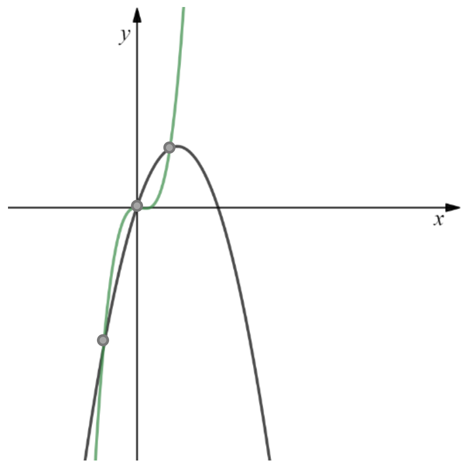
Example 2: Sketch the following functions and find the x coordinate of the intersection points.

$f(x) = 3x - x^2$ $\quad g(x) = 2x^3 - x^2$

The curves have been sketched using the methods you have learnt in this course. To find the intersection point, we must obtain the x coordinate of the locations. To solve this, we need to find the solutions to

- $f(x) = g(x)$

$$3x - x^2 = 2x^3 - x^2$$
$$2x^3 - 3x = 0$$
$$x(2x^2 - 3) = 0$$
$$x = 0 \;\&\; 2x^2 - 3 = 0$$
$$x = 0, x = \sqrt{\frac{3}{2}} \;\&\; x = -\sqrt{\frac{3}{2}}$$



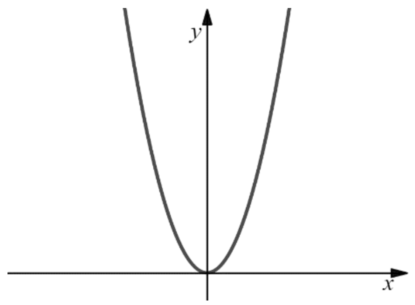
Translating graphs

Adding or subtracting a constant outside, $y = f(x) + a$, or inside, $y = f(x + a)$, a function can translate a graph vertically or horizontally respectively. Note that when translating functions, the asymptote of that function is also translated if it has one.

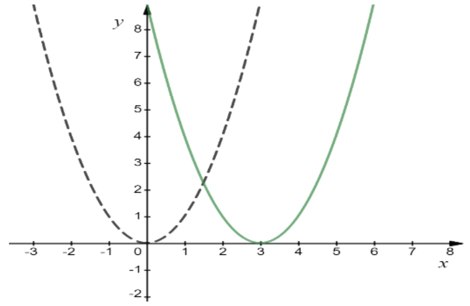
- The translation of $y = f(x) + a$ can be represented by the vector $\binom{0}{a}$
- The translation of $y = f(x + a)$ can be represented by the vector $\binom{-a}{0}$

Example 3: Given that $f(x) = x^2$, sketch the curve of $y = f(x - 3)$.

This is the graph of $y = x^2$



Applying the translation of $y = f(x - 3)$ would shift the graph by three units to the right, forming a new graph of $y = (x - 3)^2$



Stretching Graphs

Similar to translating graphs, multiplying a constant outside, $y = af(x)$, or inside, $y = f(ax)$, a function stretches the graph in the vertical direction or horizontal direction respectively.

- $y = af(x)$ would stretch the graph in the vertical direction by a multiple of a.
- $y = f(ax)$ would stretch the graph in the horizontal direction by a multiple of $\frac{1}{a}$.
- $y = -f(x)$ would be the reflection of $y = f(x)$ in the x-axis.
- $y = f(-x)$ would be the reflection of $y = f(x)$ in the y-axis.

Example 4: Given that $f(x) = 16 - 4x^2$, sketch the curve with the equation $y = \frac{1}{2}f(x)$.

Before we start to sketch $y = \frac{1}{2}f(x)$, we need to know the x and y intercepts of the curve $y = f(x)$ and how the curve looks like.

$$y = 16 - 4x^2$$
$$y = (4 - 2x)(4 + 2x)$$

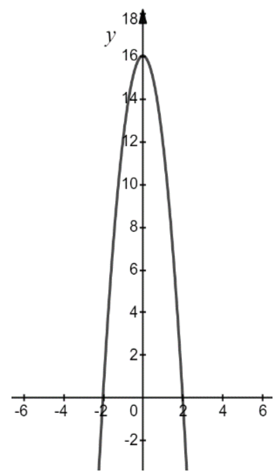
To find the x intercept we need to equate y as 0.

$$0 = (4 - 2x)(4 + 2x)$$
$$x = 2 \text{ and } x = -2$$

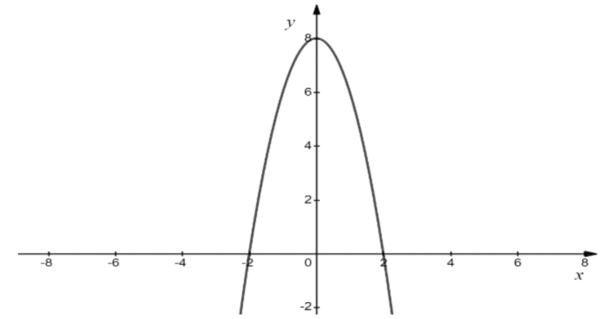
To find the y intercept we need to substitute x as $0 \therefore x = 0$.

$$y = 16 - 3 \times 0^2$$
$$y = 16$$

Hence, the curve of $y = f(x)$ is:



The transformation of $y = \frac{1}{2}f(x)$ would stretch the graph by a multiple of ½. Hence the new y intercept would be at $(16 \times \frac{1}{2}, 0) \rightarrow (8,0)$. The x intercept would not change as this transformation only effects the vertical direction.

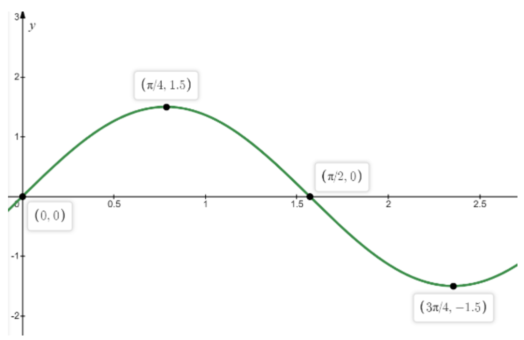


Transforming graphs

You may come across graphs with functions that are difficult to recognise. You can still apply various transformations to these types of functions by using key points such as intersection points, turning points and the x & y intercepts.

Example 5: The graph of $y = f(x)$ is given. Sketch the graph of $y = -f(x)$

The graph of $y = f(x)$



The graph of $y = -f(x)$ would be the reflection of $y = f(x)$ in the x-axis.