Algebraic Methods Cheat Sheet

In this chapter you will learn about Algebraic fractions and constructing mathematical proofs

Algebraic Fractions:

Fractions whose numerator and denominator are algebraic expressions are called algebraic fractions

Simplifying algebraic fractions:

To simplify algebraic fractions you will have to cancel common factor. But sometimes, you have to factorise the expression before you cancel common factor.

Example 1:

a)

$$\frac{8x^4-4x^3+6x}{2x}$$

Divide each numerator by $2x$

$$= \frac{8x^4}{2x} - \frac{4x^3}{2x} + \frac{6x}{2x}$$

$$= 4x^3 - 2x^2 + 3$$

b)

$$\frac{(x+4)(3x-1)}{(3x-1)}$$

Cancel the common factor of $(3x-1)$

$$= \ x+4$$

With Factorise

C)

Factorise — Cancel Common Factor

$$\frac{x^2+3x+2}{x^2+5x+4} = \frac{(x+2)(x+1)}{(x+4)(x+1)} = \frac{x+2}{x+4}$$

Dividing polynomials

A polynomial is a finite expression with positive whole number indices (≥ 0)

Polynomials	Not polynomials
$3x+5, 3x^2y+5y+6, 8$	- $\sqrt{x}$, $5x^{-2}$, $\frac{4}{x}$

You can use long division to divide polynomial by $(x \pm p)$, where p is a constant

Example 2: Write the polynomial $4x^3+9x^2-3x-10$ in the form $(x \pm p)(ax^2+bx+c)$ by dividing

$$\begin{array}{r} 4x^2 \\ x+2\overline{)4x^3+9x^2-3x-10} \\ 4x^3+8x^2 \\ \hline x^2-3x \end{array}$$

Start by dividing the first term by x, so that $4x^3 \div x = 4x^2$

Multiply $(x+2)$ by $4x^2$
So that $4x^2 \times (x+2) = 4x^3+8x^2$

Subtract,
So that $(4x^3+9x^2)-(4x^3+8x^2) = x^2$
And copy $-3x$

$$\begin{array}{r} 4x^2+x-5 \\ x+2\overline{)4x^3+9x^2-3x-10} \\ 4x^3+8x^2 \\ \hline x^2-3x \\ x^2+2x \\ \hline -5x-10 \\ -5x-10 \\ \hline 0 \end{array}$$

Repeat the process till you get a remainder

If the remainder is 0 then the divisor, in this case $(x+2)$ is a factor of polynomial $4x^3+9x^2-3x-10$

Hence, $4x^3+9x^2-3x-10 = (x+2)(4x^2+x-5)$

The factor theorem:

The factor theorem is a quick way of finding simple linear factor of a polynomial

The factor theorem states that if f(x) is a polynomial then,

- If f(p) = 0, then $(x-p)$ is a factor of f(x)
- If $(x-p)$ is a factor of f(x) , then f(p) = 0

Example 3: f$(x) = 3x^3-12x^2+6x-24$

a) Use factor theorem to show that $(x-4)$ is a factor of f(x)
b) Hence, show that 4 is the only real root of the equation f$(x) = 0$

a)

According to the theorem,

If $(x-4)$ is a factor of $3x^3-12x^2+6x-24$, then f(4) must be equal to 0

Substitute $x = 4$ in the polynomial

$$f(x) = 3x^3-12x^2+6x-24$$
$$\therefore f(4) = 3(4)^3 - 12(4)^2 + 6(4) - 24$$
$$= 192 - 192 + 24 - 24$$
$$= 0$$

So $(x-4)$ is a factor of $3x^3-12x^2+6x-24$

b)

To find the root of the equation, first you need to use long division to factorise the polynomial and equate it to 0

$$\begin{array}{r} 3x^2+6 \\ x-4\overline{)3x^3-12x^2+6x-24} \\ 3x^3-12x^2 \\ \hline 6x-24 \\ 6x-24 \\ \hline 0 \end{array}$$

$f(x) = (x-4)(3x^2+6)$

Equate $f(x) = 0$

$(x-4)(3x^2+6) = 0$

$3x^2+6$ is a quadratic equation

$\Rightarrow \ a = 3, b = 0, c = 6$

And to check if the roots are real or not, you need to find discriminant i.e. b^2-4ac

If $b^2-4ac < 0 \Rightarrow$ equation has no real roots

By substituting the values of a, b and c in the discriminant we get,

$b^2-4ac = 0-4(3)(6) = -72 < 0$

Hence $3x^2+6$ has no real roots. Therefore $f(x)$ has only one real root of $x = 4$

Mathematical proof:

Key terms:

Theorem	Mathematical statement (or a Conjecture)
A statement that has been proven	A statement that has yet to be proven

In this section you will have to prove mathematical statement (or conjecture). In simple words you will have to show that the mathematical statement is true in specified cases. You will have to use the following steps to prove a statement

Start with know facts or theorems → Clearly show logical steps → Statement of proof

Example 4: Prove that n^2-n is an even number for all values of n.

You know the fact that ODD×EVEN = EVEN.

Start by writing n^2-n as multiple of two terms. We can do that by factorising the term as follows

$n^2-n = n(n-1)$

Any number is either ODD or EVEN. Now consider if n is even, then $n-1$ must be odd which implies

$n \times (n-1) \Rightarrow$ Even × Odd = Even

If $n-1$ is even then n must be odd which implies

$n \times (n-1) \Rightarrow$ Odd × Even = Even

Hence, n^2-n is even for all values of n

Prove an Identity:

Identical statements mean they are always equal mathematically. In this section you will have to prove an identity. That is, you will have to show the right hand side of the equation equal to left hand side.

Example 5: Prove that $(x+\sqrt{y})(x-\sqrt{y}) \equiv x^2-y$

Start by solving one side of the identity. It will be logical to start with $(x+\sqrt{y})(x-\sqrt{y})$ as this can be expanded.

$$(x+\sqrt{y})(x-\sqrt{y}) = x(x-\sqrt{y}) + \sqrt{y}(x-\sqrt{y})$$
$$= x^2 - x\sqrt{y} + x\sqrt{y} - (\sqrt{y}\sqrt{y}) \qquad as \sqrt{y} \times \sqrt{y} = y$$
$$= x^2 - y$$

$\Rightarrow (x+\sqrt{y})(x-\sqrt{y}) \equiv x^2-y$

Hence, we have proved the identity.

Methods of proof:

There are different methods to prove a mathematical statement. However, in this chapter you will only learn Proof by Exhaustion.

Proof by Exhaustion: In this method you will have to split your statement into smaller cases and prove each case separately. This way you will be able to prove that the statement is true.

Example 6: Prove that the sum of two consecutive square numbers between 1^2 and 8^2 is an odd number.

You will prove this by exhaustion. Start by listing all square numbers between 1^2 and 8^2 and add the consecutive square numbers to get a result,

$2^2+3^2 = Odd, 3^2+4^2 = Odd, 4^2+5^2 = odd, 5^2+6^2 = odd, 6^2+7^2 = Odd$

Now you can see, each case is proved to be an odd number

So, the sum of two consecutive square numbers between 1^2 and 8^2 is always an odd number.

Counter-example:

You can prove a mathematical statement is not true by counter-example. A counter-example is one example that does not work for the given statement. To disprove a statement one counter example is enough.

Example 7: Show, by means of a counter-example, that the following inequality does not hold when p and q are both negative

$$p+q > \sqrt{4pq}$$

Start by taking negative values for both p and q

$p = -1, q = -2$

$p+q = (-1)-(-2) = -1+2 = 1$

$\sqrt{4pq} = \sqrt{4(-1)(-2)} = \sqrt{8}$

But $1 < \sqrt{8}$, i.e. $p+q < \sqrt{4pq}$

Hence by counter example, we proved the inequality is not true for negative values