Oxford Cambridge and RSA

A Level Mathematics A

H240/03 Pure Mathematics and Mechanics

Friday 15 June 2018 – Afternoon

Time allowed: 2 hours

You must have:
- Printed Answer Booklet

You may use:
- a scientific or graphical calculator

Model Solutions

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INSTRUCTIONS
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Complete the boxes provided on the Printed Answer Booklet with your name, centre number and candidate number.
- Answer **all** the questions.
- **Write your answer to each question in the space provided in the Printed Answer Booklet.** If additional space is required, you should use the lined page(s) at the end of the Printed Answer Booklet. The question number(s) must be clearly shown.
- Do **not** write in the barcodes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The acceleration due to gravity is denoted by $g\,\text{m}\,\text{s}^{-2}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION
- The total mark for this paper is **100**.
- The marks for each question are shown in brackets **[ ]**.
- **You are reminded of the need for clear presentation in your answers.**
- The Printed Answer Booklet consists of **16** pages. The Question Paper consists of **8** pages.

Formulae
A Level Mathematics A (H240)

Arithmetic series

$S_n = \frac{1}{2}n(a+l) = \frac{1}{2}n\{2a+(n-1)d\}$

Geometric series

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_\infty = \frac{a}{1-r} \quad \text{for } |r| < 1$$

Binomial series

$$(a+b)^n = a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \ldots + {}^nC_r a^{n-r}b^r + \ldots + b^n \qquad (n \in \mathbb{N}),$$

where ${}^nC_r = {}_nC_r = \binom{n}{r} = \dfrac{n!}{r!(n-r)!}$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots \qquad (|x| < 1,\ n \in \mathbb{R})$$

Differentiation

$\mathrm{f}(x)$	$\mathrm{f}'(x)$
$\tan kx$	$k\sec^2 kx$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$

Quotient rule $y = \dfrac{u}{v}$, $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

Differentiation from first principles

$$\mathrm{f}'(x) = \lim_{h \to 0} \frac{\mathrm{f}(x+h) - \mathrm{f}(x)}{h}$$

Integration

$$\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,dx = \ln|\mathrm{f}(x)| + c$$

$$\int \mathrm{f}'(x)(\mathrm{f}(x))^n\,dx = \frac{1}{n+1}(\mathrm{f}(x))^{n+1} + c$$

Integration by parts $\displaystyle\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$

Small angle approximations

$\sin\theta \approx \theta$, $\cos\theta \approx 1 - \frac{1}{2}\theta^2$, $\tan\theta \approx \theta$ where θ is measured in radians

Trigonometric identities

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \qquad \left(A \pm B \neq (k + \tfrac{1}{2})\pi\right)$

Numerical methods

Trapezium rule: $\int_a^b y \, dx \approx \frac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\}$, where $h = \dfrac{b-a}{n}$

The Newton-Raphson iteration for solving $f(x) = 0$: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

Probability

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cap B) = P(A)P(B \mid A) = P(B)P(A \mid B)$ **or** $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$

Standard deviation

$\sqrt{\dfrac{\Sigma(x-\bar{x})^2}{n}} = \sqrt{\dfrac{\Sigma x^2}{n} - \bar{x}^2}$ **or** $\sqrt{\dfrac{\Sigma f(x-\bar{x})^2}{\Sigma f}} = \sqrt{\dfrac{\Sigma fx^2}{\Sigma f} - \bar{x}^2}$

The binomial distribution

If $X \sim B(n, p)$ then $P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$, Mean of X is np, Variance of X is $np(1-p)$

Hypothesis test for the mean of a normal distribution

If $X \sim N(\mu, \sigma^2)$ then $\bar{X} \sim N\left(\mu, \dfrac{\sigma^2}{n}\right)$ and $\dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$

Percentage points of the normal distribution

If Z has a normal distribution with mean 0 and variance 1 then, for each value of p, the table gives the value of z such that $P(Z \leqslant z) = p$.

p	0.75	0.90	0.95	0.975	0.99	0.995	0.9975	0.999	0.9995
z	0.674	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291

Kinematics

Motion in a straight line

$v = u + at$

$s = ut + \frac{1}{2}at^2$

$s = \frac{1}{2}(u + v)t$

$v^2 = u^2 + 2as$

$s = vt - \frac{1}{2}at^2$

Motion in two dimensions

$\mathbf{v} = \mathbf{u} + \mathbf{a}t$

$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$

$\mathbf{s} = \frac{1}{2}(\mathbf{u} + \mathbf{v})t$

$\mathbf{s} = \mathbf{v}t - \frac{1}{2}\mathbf{a}t^2$

Section A: Pure Mathematics

Answer **all** the questions.

1 A circle with centre C has equation $x^2 + y^2 + 8x - 2y - 7 = 0$.

Find

(i) the coordinates of C, **[2]**

(ii) the radius of the circle. **[1]**

1 i) $x^2 + y^2 + 8x - 2y - 7 = 0$

$(x+4)^2 + (y-1)^2 - 16 - 1 - 7 = 0$

$(x+4)^2 + (y-1)^2 = 24$

Radius : $\sqrt{24}$

C : $(-4, 1)$

2 Solve the equation $|2x-1| = |x+3|$. **[3]**

2 $|2x-1| = |x+3|$

Either $2x - 1 = x + 3$ or $-2x + 1 = x + 3$

$x = 4$ $\quad$ $-2 = 3x$

$x = -\frac{2}{3}$

3 **In this question you must show detailed reasoning.**

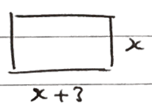
A gardener is planning the design for a rectangular flower bed. The requirements are:

- the length of the flower bed is to be 3 m longer than the width,
- the length of the flower bed must be at least 14.5 m,
- the area of the flower bed must be less than 180 m^2.

The width of the flower bed is x m.

By writing down and solving appropriate inequalities in x, determine the set of possible values for the width of the flower bed. **[6]**

3.



$x + 3 \geq 14.5$

$x \geq 11.5$

Area $= x(x+3) < 180$

$x^2 + 3x - 180 < 0$

$(x-12)(x+15) < 0$

So we know $x \geq 11.5$ and $-15 < x < 12$

Putting these together you get $11.5 \leq x < 12$

4 In this question you must show detailed reasoning.

The functions f and g are defined for all real values of x by

$$\mathrm{f}(x) = x^3 \quad \text{and} \quad \mathrm{g}(x) = x^2 + 2.$$

(i) Write down expressions for

(a) $\mathrm{fg}(x)$, **[1]**

4 i) $f(x) = x^3$ $\quad g(x) = x^2 + 2$

a) $fg(x) = f(x^2+2) = (x^2+2)^3$

(b) $\mathrm{gf}(x)$. **[1]**

b) $gf(x) = g(x^3) = (x^3)^2 + 2$

$= x^6 + 2$

(ii) Hence find the values of x for which $\mathrm{fg}(x) - \mathrm{gf}(x) = 24$. **[6]**

ii) $fg(x) - gf(x) = 24$

$(x^2+2)^3 - (x^6+2) = 24$

$(x^2+2)(x^4+4x^2+4) - x^6 - 2 = 24$

$\cancel{x^6} + 4x^4 + 4x^2 + 2x^4 + 8x^2 + 8 - \cancel{x^6} = 26$

$6x^4 + 12x^2 - 18 = 0$

$x^4 + 2x^2 - 3 = 0$

$(x^2-1)(x^2+3) = 0$

$x^2 + 3 = 0$ has no solutions

$x^2 - 1 = 0$

$x^2 = 1$

$x = \pm 1$

5 **(i)** Use the trapezium rule, with two strips of equal width, to show that

$$\int_0^4 \frac{1}{2+\sqrt{x}}\,dx \approx \frac{11}{4} - \sqrt{2}.$$ **[5]**

5 i) $h = \frac{4-0}{2} = 2$, let $f(x) = \frac{1}{2+\sqrt{x}}$

$I = \frac{h}{2}\left[f(0) + f(4) + 2f(2)\right]$

$= \frac{2}{2}\left[\frac{1}{2} + \frac{1}{4} + 2\left(\frac{1}{2+\sqrt{2}}\right)\right]$

$= \frac{3}{4} + \frac{2}{2+\sqrt{2}}$

$= \frac{3}{4} + \frac{2}{2+\sqrt{2}} \times \frac{2-\sqrt{2}}{2-\sqrt{2}}$

$= \frac{3}{4} + \frac{4-2\sqrt{2}}{4-2}$

$= \frac{3}{4} + 2 - \sqrt{2}$

$= \frac{11}{4} - \sqrt{2}$

(ii) Use the substitution $x = u^2$ to find the exact value of

$$\int_0^4 \frac{1}{2+\sqrt{x}}\,dx.$$ **[6]**

ii) $x = u^2$

$\frac{dx}{du} = 2u \Rightarrow dx = 2u\ du$

when $x = 4,\ u = 2$
$x = 0,\ u = 0$

$\int_0^4 \frac{1}{2+\sqrt{x}}\,dx = \int_0^2 \frac{1}{2+u} \cdot 2u\ du$

$= 2\int \frac{u}{2+u}\,du$

$= 2\int 1 - \frac{2}{2+u}\,du$

$= \left[u - 2\ln(2+u) \right]_0^2$

$= 2\left[2 - 2\ln 4 - 0 + 2\ln 2\right]$

$= 2(2 - 4\ln 2 + 2\ln 2)$

$= 2(2 - 2\ln 2)$

(iii) Using your answers to parts **(i)** and **(ii)**, show that

$$\ln 2 \approx k + \frac{\sqrt{2}}{4},$$

where k is a rational number to be determined. **[2]**

iii) $\frac{11}{4} - \sqrt{2} = 2(2 - 2\ln 2)$

$\frac{11}{4} - \sqrt{2} = 4 - 4\ln 2$

$4\ln 2 = \frac{5}{4} + \sqrt{2}$

$\ln 2 = \frac{5}{16} + \frac{\sqrt{2}}{4}$

$k = \frac{5}{16}$

6 It is given that the angle θ satisfies the equation $\sin\left(2\theta+\frac{1}{4}\pi\right)=3\cos\left(2\theta+\frac{1}{4}\pi\right)$.

(i) Show that $\tan 2\theta=\frac{1}{2}$. **[3]**

6 i) $\sin\left(2\theta+\frac{1}{4}\pi\right) = 3\cos\left(2\theta+\frac{1}{4}\pi\right)$

$\sin 2\theta \cos\frac{\pi}{4} + \sin\frac{\pi}{4}\cos 2\theta = 3\cos 2\theta\cos\frac{\pi}{4} - 3\sin 2\theta \sin\frac{\pi}{4}$

$\cancel{\frac{\sqrt{2}}{2}}\sin 2\theta + \cancel{\frac{\sqrt{2}}{2}}\cos 2\theta = \frac{3\cancel{\sqrt{2}}}{\cancel{2}}\cos 2\theta - \frac{3\cancel{\sqrt{2}}}{\cancel{2}}\sin 2\theta$

$4\sin 2\theta = 2\cos 2\theta$

$\frac{\sin 2\theta}{\cos 2\theta} = \frac{2}{4}$

$\tan 2\theta = \frac{1}{2}$

(ii) Hence find, in surd form, the exact value of $\tan\theta$, given that θ is an obtuse angle. **[5]**

ii) $\tan 2\theta = \frac{1}{2}$

$\frac{2\tan\theta}{1-\tan^2\theta} = \frac{1}{2}$

$4\tan\theta = 1-\tan^2\theta$

$\tan^2\theta + 4\tan\theta - 1 = 0$

$\tan\theta = \frac{-4 \pm \sqrt{4^2-4(-1)}}{2}$

$\tan\theta = \frac{-4\pm\sqrt{20}}{2}$

$\tan\theta = -2\pm\sqrt{5}$

$-2+\sqrt{5} > 0$ so gives an acute angle

$\therefore \tan\theta = -2-\sqrt{5}$

7 The gradient of the curve $y = f(x)$ is given by the differential equation

$$(2x-1)^3 \frac{dy}{dx} + 4y^2 = 0$$

and the curve passes through the point $(1,\ 1)$. By solving this differential equation show that

$$f(x) = \frac{ax^2 - ax + 1}{bx^2 - bx + 1},$$

where a and b are integers to be determined. **[9]**

7. $(2x-1)^3 \frac{dy}{dx} + 4y^2 = 0$

$(2x-1)^3 \frac{dy}{dx} = -4y^2$

$\int \frac{-1}{4y^2}\, dy = \int \frac{1}{(2x-1)^3}\, dx$

$\frac{1}{4y} = \frac{-1}{4} \cdot (2x-1)^{-2} + c$

$\frac{1}{y} = -\frac{1}{(2x-1)^2} + c$

At $x = 1$, $y = 1$: $\frac{1}{1} = \frac{-1}{(2-1)^2} + c$

$1 = -1 + c$

$c = 2$

$\frac{1}{y} = \frac{-1}{(2x-1)^2} + 2$

$y = \frac{1}{\frac{-1}{(2x-1)^2} + 2}$

$y = \frac{(2x-1)^2}{-1 + 2(2x-1)^2}$

$$y = \frac{4x^2 - 4x + 1}{-1 + 8x^2 - 8x + 2}$$

$$y = \frac{4x^2 - 4x + 1}{8x^2 - 8x + 1}$$

Section B: Mechanics

Answer **all** the questions.

8 In this question $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ denote unit vectors which are horizontal and vertically upwards respectively.

A particle of mass 5 kg, initially at rest at the point with position vector $\begin{pmatrix}2\\45\end{pmatrix}$ m, is acted on by gravity and also by two forces $\begin{pmatrix}15\\-8\end{pmatrix}$ N and $\begin{pmatrix}-7\\-2\end{pmatrix}$ N.

(i) Find the acceleration vector of the particle. **[3]**

8 i) $\begin{pmatrix}15\\-8\end{pmatrix} + \begin{pmatrix}-7\\-2\end{pmatrix} + 5\begin{pmatrix}0\\-9.8\end{pmatrix} = 5\underline{a}$

$\begin{pmatrix}8\\-10\end{pmatrix} + \begin{pmatrix}0\\-49\end{pmatrix} = 5\underline{a}$

$5\underline{a} = \begin{pmatrix}8\\-59\end{pmatrix}$

$\underline{a} = \begin{pmatrix}1.6\\-11.8\end{pmatrix}$

(ii) Find the position vector of the particle after 10 seconds. **[3]**

ii) $s = s$

$u = 0$

$v = -$

$a = \begin{pmatrix}1.6\\-11.8\end{pmatrix}$

$t = 10$

$s = ut + \frac{1}{2}at^2$

$s = \frac{1}{2}(10)^2\begin{pmatrix}1.6\\-11.8\end{pmatrix}$

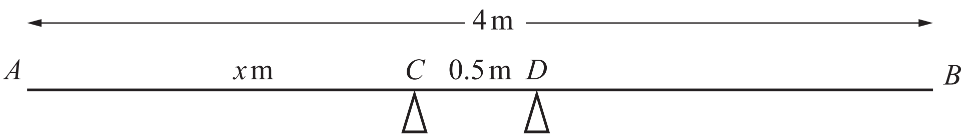
$s = 50\begin{pmatrix}1.6\\-11.8\end{pmatrix}$

$s = \begin{pmatrix}80\\-590\end{pmatrix}$

This is how far the particle has moved, but we need to add on its inital starting position

$$\begin{pmatrix} 80 \\ -590 \end{pmatrix} + \begin{pmatrix} 2 \\ 45 \end{pmatrix} = \begin{pmatrix} 82 \\ -545 \end{pmatrix}$$

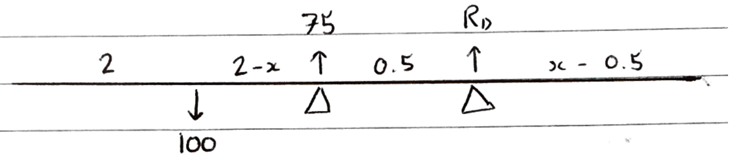
9 A uniform plank AB has weight 100 N and length 4 m. The plank rests horizontally in equilibrium on two smooth supports C and D, where $AC = x$ m and $CD = 0.5$ m (see diagram).



The magnitude of the reaction of the support on the plank at C is 75 N. Modelling the plank as a rigid rod, find

(i) the magnitude of the reaction of the support on the plank at D, **[1]**

9 i)



$R(\uparrow)$: $75 + R_D = 100$

$R_D = 25$

(ii) the value of x. **[3]**

ii) $\circlearrowleft C$: $100(2 - x) = R_D(0.5)$

$200 - 100x = 25 \times 0.5$

$100x = 200 - 12.5$

$100x = 187.5$

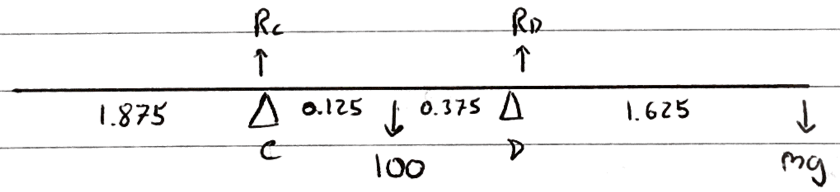
$x = 1.875$

[1]

A stone block, which is modelled as a particle, is now placed at the end of the plank at *B* and the plank is on the point of tilting about *D*.

(iii) Find the weight of the stone block. **[3]**

iii



The plank is on the point of tilting about D, meaning $R_C = 0$

$\circlearrowright D$: $100 \times 0.375 = 1.625m$

$m = 23.1$ N

(iv) Explain the limitation of modelling

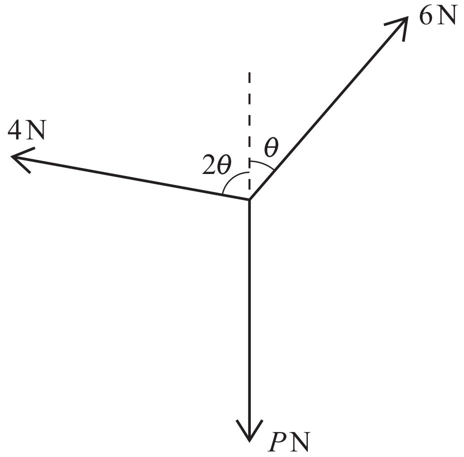
(a) the stone block as a particle, **[1]**

(b) the plank as a rigid rod. **[1]**

iv) The block has dimensions so its weight is not focused at exactly B, which is what you assume when it is a particle

You are assuming the rod does not bend

10 Three forces, of magnitudes 4 N, 6 N and P N, act at a point in the directions shown in the diagram.



The forces are in equilibrium.

(i) Show that $\theta = 41.4°$, correct to 3 significant figures. **[4]**

10 i) $R(\rightarrow)$: $4\sin 2\theta = 6\sin\theta$

$8\cos\theta\cancel{\sin\theta} = 6\cancel{\sin\theta}$

$\cos\theta = \frac{3}{4}$

$\theta = 41.4096\ldots$

$\theta = 41.4$

(ii) Hence find the value of P. **[2]**

ii) $R(\uparrow)$: $P = 4\cos 2\theta + 6\cos\theta$

$P = 0.5 + 4.5$

$P = 5$

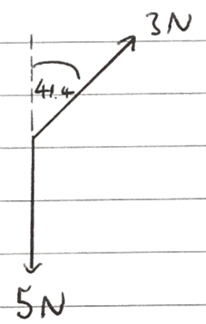
The force of magnitude 4 N is now removed and the force of magnitude 6 N is replaced by a force of magnitude 3 N acting in the same direction.

(iii) Find

(a) the magnitude of the resultant of the two remaining forces, **[3]**

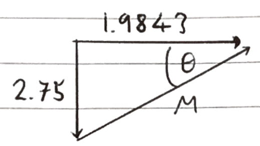
(b) the direction of the resultant of the two remaining forces. **[2]**

iii)



$\rightarrow$: $3\sin 41.4 = 1.9843$

$\downarrow$: $5 - 3\cos 41.4 = 2.75$



$M = \sqrt{2.75^2 + 1.9843^2} = \underline{3.39\text{ N}}$

$\tan\theta = \dfrac{2.75}{1.9843}$

$\theta = 54.18°$ below the horizontal

11 The velocity $v\,\mathrm{m\,s^{-1}}$ of a car at time t s, during the first 20 s of its journey, is given by $v = kt + 0.03t^2$, where k is a constant. When $t = 20$ the acceleration of the car is $1.3\,\mathrm{m\,s^{-2}}$. For $t > 20$ the car continues its journey with constant acceleration $1.3\,\mathrm{m\,s^{-2}}$ until its speed reaches $25\,\mathrm{m\,s^{-1}}$.

(i) Find the value of k. **[3]**

11 i) $v = kt + 0.03t^2$

$a = \frac{dv}{dt} = k + 0.06t$

At $t = 20$, $a = 1.3$

$1.3 = k + 0.06(20)$

$1.3 - 1.2 = k$

$\underline{k = 0.1}$

(ii) Find the total distance the car has travelled when its speed reaches $25\,\mathrm{m\,s^{-1}}$. **[7]**

ii) $s = \int v = \int 0.1t + 0.03t^2 \, dt$

$= 0.05t + 0.01t^3 + c$

when $t = 0$, $s = 0 \Rightarrow c = 0$

$s = 0.05t + 0.01t^3$

In the first 20 seconds he travels

$s = 0.05(20) + 0.01(20)^3$

$s = 100$

For $t > 20$: $s = s$

$u = 0.1(20) + 0.03(20)^2 = 14$

$v = 25$

$a = 1.3$

$t = -$

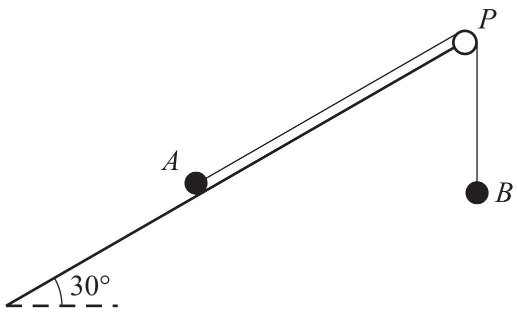
$v^2 = u^2 + 2as$

$25^2 = 14^2 + 2(1.3)s$

$429 = 2.6s$

$s = 165$

12 One end of a light inextensible string is attached to a particle A of mass m kg. The other end of the string is attached to a second particle B of mass λm kg, where λ is a constant. Particle A is in contact with a rough plane inclined at 30° to the horizontal. The string is taut and passes over a small smooth pulley P at the top of the plane. The part of the string from A to P is parallel to a line of greatest slope of the plane. The particle B hangs freely below P (see diagram).

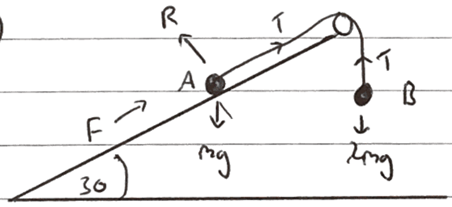


The coefficient of friction between A and the plane is μ.

(i) It is given that A is on the point of moving down the plane.

(a) Find the exact value of μ when $\lambda = \frac{1}{4}$. **[7]**

12 i) a)



$R = mg\cos 30 = \frac{\sqrt{3}}{2}mg$

$F = \mu R = \frac{\sqrt{3}}{2}mg\mu$

B : $\lambda mg = T$

$\frac{1}{4}mg = T$ ①

A : $T + F = mg\sin 30$

$T + \frac{\sqrt{3}}{2}mg\mu = \frac{1}{2}mg$ ②

Sub ① into ② :

$$\frac{1}{4}\cancel{m}g + \frac{\sqrt{3}}{2}\cancel{m}g\mu = \frac{1}{2}\cancel{m}g$$

$$\frac{\sqrt{3}}{2}\mu = \frac{1}{4}$$

$$\mu = \frac{1}{2\sqrt{3}}$$

$$\mu = \sqrt{3}$$

(b) Show that the value of λ must be less than $\frac{1}{2}$. **[2]**

b) The equation of motion for B is still
$T = \lambda mg$
Sub this into the equation for A

$$T + F = mg\sin 30$$
$$\lambda mg + F = \frac{1}{2}mg$$
$$F = \frac{1}{2}mg - \lambda mg$$

We know $F > 0$ because it is on the point of moving ~~up~~ down the plane so friction must be acting up

$$F > 0$$
$$\frac{1}{2}mg - \lambda mg > 0$$
$$\frac{1}{2}\cancel{mg} > \lambda\cancel{mg}$$
$$\lambda < \frac{1}{2}$$

(ii) Given instead that $\lambda = 2$ and that the acceleration of A is $\frac{1}{4}g\,\mathrm{m\,s^{-2}}$, find the exact value of μ. **[5]**

ii) As before, $R = mg\cos 30 = \frac{\sqrt{3}}{2}mg$

$F = \mu R = \mu \times \frac{\sqrt{3}}{2}mg = \frac{\sqrt{3}}{2}mg\mu$

Find the equations of motion of A and B

B : $2mg - T = 2m(\frac{1}{4}g)$

$2mg - T = \frac{1}{2}mg$

$T = \frac{3}{2}mg$ ①

A : $T - F - mg\sin 30 = m(\frac{1}{4}g)$

$T - \frac{\sqrt{3}}{2}mg\mu - \frac{1}{2}mg = \frac{1}{4}mg$

sub in ①:

$\frac{3}{2}mg - \frac{\sqrt{3}}{2}mg\mu - \frac{1}{2}mg = \frac{1}{4}mg$

$\frac{3}{4}mg = \frac{\sqrt{3}}{2}mg\mu$

$\mu = \frac{3}{2\sqrt{3}}$

$\mu = \frac{\sqrt{3}}{2}$

END OF QUESTION PAPER