MODEL ANSWERS

OCR
Oxford Cambridge and RSA

Wednesday 14 October 2020 – Afternoon

AS Level Mathematics A

H230/02 Pure Mathematics and Mechanics

Time allowed: 1 hour 30 minutes

You must have:
- the Printed Answer Booklet
- a scientific or graphical calculator

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INSTRUCTIONS

- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question.
- The acceleration due to gravity is denoted by $g\,\text{m}\,\text{s}^{-2}$. When a numerical value is needed use $g = 9.8$ unless a different value is specified in the question.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION

- The total number of marks for this paper is **75**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **8** pages.

ADVICE

- Read each question carefully before you start your answer.

Formulae
AS Level Mathematics A (H230)

Binomial series

$$(a+b)^n = a^n + {}^n\mathrm{C}_1 a^{n-1}b + {}^n\mathrm{C}_2 a^{n-2}b^2 + \ldots + {}^n\mathrm{C}_r a^{n-r}b^r + \ldots + b^n \qquad (n \in \mathbb{N}),$$

where ${}^n\mathrm{C}_r = {}_n\mathrm{C}_r = \binom{n}{r} = \dfrac{n!}{r!(n-r)!}$

Differentiation from first principles

$$\mathrm{f}'(x) = \lim_{h \to 0} \frac{\mathrm{f}(x+h) - \mathrm{f}(x)}{h}$$

Standard deviation

$$\sqrt{\frac{\Sigma(x-\bar{x})^2}{n}} = \sqrt{\frac{\Sigma x^2}{n} - \bar{x}^2} \text{ or } \sqrt{\frac{\Sigma f(x-\bar{x})^2}{\Sigma f}} = \sqrt{\frac{\Sigma fx^2}{\Sigma f} - \bar{x}^2}$$

The binomial distribution

If $X \sim \mathrm{B}(n, p)$ then $\mathrm{P}(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$, mean of X is np, variance of X is $np(1-p)$

Kinematics

$v = u + at$

$s = ut + \frac{1}{2}at^2$

$s = \frac{1}{2}(u+v)t$

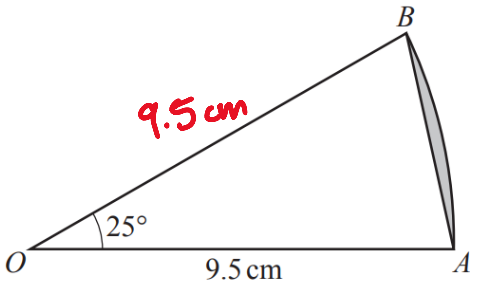
$v^2 = u^2 + 2as$

$s = vt - \frac{1}{2}at^2$

Section A: Pure Mathematics

Answer **all** the questions.

1



The diagram shows a sector AOB of a circle with centre O and radius 9.5 cm. The angle AOB is 25°.

(a) Calculate the length of the straight line AB. **[2]**

(b) Find the area of the segment shaded in the diagram. **[3]**

(a.) $a^2 = b^2 + c^2 - 2bc\cos A$

$AB^2 = 9.5^2 + 9.5^2 - 2(9.5)(9.5)\cos 25°$

$AB^2 = 180.5 - 163.58...$

$AB^2 = 16.911...$

$\therefore AB^2 = 4.112... \approx 4.11\text{ cm} \quad (3sf)$

(b.) Area of Sector $= \frac{\theta}{360}\pi r^2 = \frac{25}{360}\pi(9.5)^2 = 19.68...$

Area of Triangle $= \frac{1}{2}bc\sin A = \frac{1}{2}(9.5)(9.5)\sin 25° = 19.07...$

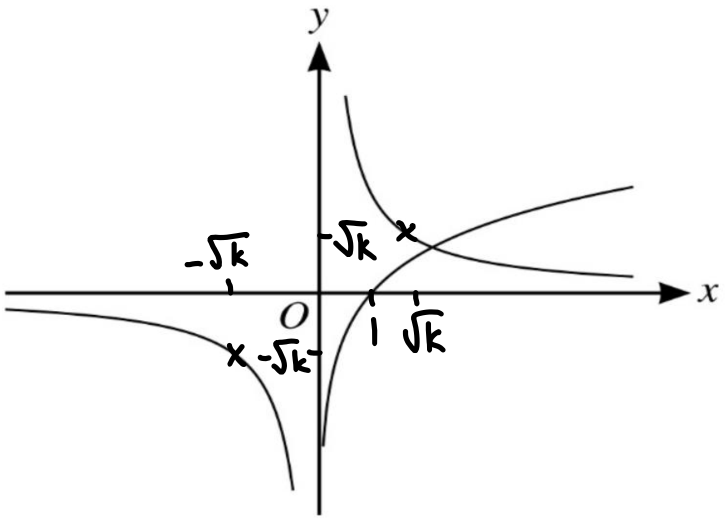
$\therefore$ Shaded Area $= 19.68... - 19.07... = 0.6188... \approx 0.619\text{ cm}^2$ (3sf)

2 Two curves have equations $y = \ln x$ and $y = \frac{k}{x}$, where k is a positive constant.

(a) Sketch the curves on a **single** diagram. **[3]**

(b) Explain how your diagram shows that the equation $x \ln x - k = 0$ has exactly one real root. **[2]**

(a.)



(b.) $\ln x = \frac{k}{x}$ can be rearranged to $x \ln x - k = 0$.

The curves intersect at a single point as shown by the graph, so there is only one real root.

3 **In this question you must show detailed reasoning.**

Find the equation of the normal to the curve $y = 4\sqrt{x} - 3x + 1$ at the point on the curve where $x = 4$. Give your answer in the form $ax + by + c = 0$, where a, b and c are integers. **[7]**

$$y = 4x^{\frac{1}{2}} - 3x + 1$$

$$\frac{dy}{dx} = \frac{4}{2}x^{-\frac{1}{2}} - 3 = \frac{2}{x^{1/2}} - 3 = \frac{2}{\sqrt{x}} - 3$$

$x = 4 \Rightarrow \frac{dy}{dx} = \frac{2}{\sqrt{4}} - 3 = -2$ → Gradient of Tangent

$\therefore$ Gradient of Normal $= \frac{1}{2}$

$x = 4 \Rightarrow y = 4\sqrt{4} - 3(4) + 1 = -3$ $\quad\therefore$ Coordinate $(4, -3)$

$\times 2 \; y - -3 = \frac{1}{\cancel{2}}(x - 4)$ $\quad \times 2 \quad \times\cancel{2}$

$$2y + 6 = x - 4$$

$$x - 4 - 2y - 6 = 0$$

$$\boxed{\therefore x - 2y - 10 = 0}$$

4 **In this question you must show detailed reasoning.**

The cubic polynomial $6x^3 + kx^2 + 57x - 20$ is denoted by $f(x)$. It is given that $(2x-1)$ is a factor of $f(x)$.

(a) Use the factor theorem to show that $k = -37$. **[2]**

(b) Using this value of k, factorise $f(x)$ completely. **[3]**

(c) **(i)** Hence find the three values of t satisfying the equation $6e^{-3t} - 37e^{-2t} + 57e^{-t} - 20 = 0$. **[2]**

(ii) Express the sum of the three values found in part **(c)(i)** as a single logarithm. **[2]**

(a.) $2x - 1 = 0 \Rightarrow x = \frac{1}{2}$

$f\left(\frac{1}{2}\right) = 6\left(\frac{1}{2}\right)^3 + k\left(\frac{1}{2}\right)^2 + 57\left(\frac{1}{2}\right) - 20 = 0$

$\frac{3}{4} + \frac{k}{4} + \frac{57}{2} - 20 = 0$

$\frac{k}{4} + \frac{37}{4} = 0$ ×4

×4 ×4

$k + 37 = 0 \Rightarrow \boxed{\therefore k = -37}$

(b.) $f(x) = 6x^3 - 37x^2 + 57x - 20$

$f(x) = (2x-1)(3x^2 - 17x + 20)$

$6x^3 - 3x^2 - 34x^2 + 17x + 40x - 20$

$\boxed{\therefore f(x) = (2x-1)(3x-5)(x-4)}$

(c.) (i) $f(x) = (2x-1)(3x-5)(x-4)$

$\therefore x = \frac{1}{2}, \frac{5}{3}, 4$

Given that $x = e^{-t}$, $e^{-t} = \frac{1}{2}, \frac{5}{3}, 4$

$\ln(e^{-t}) = -\ln\left(\frac{1}{2}\right), -\ln\left(\frac{5}{3}\right), -\ln(4)$

$$\boxed{\therefore t = -\ln\left(\frac{1}{2}\right), -\ln\left(\frac{5}{3}\right), -\ln(4)}$$

(ii) $\sum t = -\left[\ln\left(\frac{1}{2}\right) + \ln\left(\frac{5}{3}\right) + \ln(4)\right]$

$= -\left[\ln 1 - \ln 2 + \ln 5 - \ln 3 + \ln 4\right]$

$= -\left[0 + \ln\left(\frac{5 \times 4}{2 \times 3}\right)\right]$

$= -\ln\left(\frac{20}{6}\right)$

$= -\ln\left(\frac{10}{3}\right)$

$= \boxed{\ln\left(\frac{3}{10}\right)}$

5 A curve has equation $y = a(x+b)^2 + c$, where a, b and c are constants. The curve has a stationary point at $(-3, 2)$.

(a) State the values of b and c. **[2]**

When the curve is translated by $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$ the transformed curve passes through the point $(3, -18)$.

(b) Determine the value of a. **[3]**

(a.) $(-3, 2) \Rightarrow \boxed{b = 3, c = 2}$

(b.) Translated Curve: $y = a(x + (b-4))^2 + c$

$-18 = a(3 + (b-4))^2 + c$

$-18 = a(3 + 3 - 4)^2 + \cancel{2}$

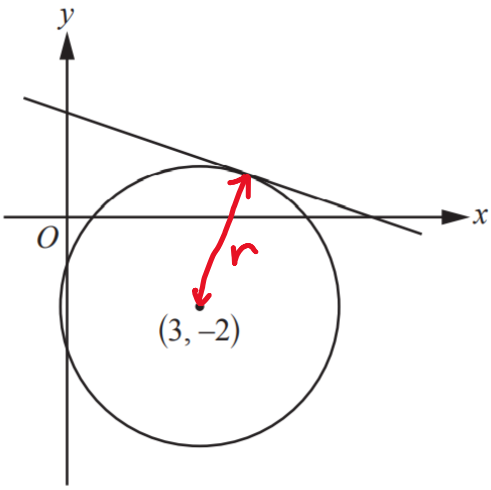
$-2 \qquad -\cancel{2}$

$-20 = a(2)^2$

$\dfrac{-20}{4} = \dfrac{\cancel{4}a}{\cancel{4}}$

$\boxed{\therefore a = -5}$

6 In this question you must show detailed reasoning.



The diagram shows the line $3y+x=7$ which is a tangent to a circle with centre $(3, -2)$.

Find an equation for the circle. **[6]**

$x^2+y^2=r^2$

$(x-3)^2+(y+2)^2=r^2$

$3y+x=7 \Rightarrow y=\frac{-x}{3}+\frac{7}{3} \Rightarrow m=-\frac{1}{3} \Rightarrow \therefore m_N=3$

①

$y+2=3(x-3)$

$y+2=3x-9$

$3x-9-y-2=0$

$\therefore 3x-y=11$ ②

Solving ① & ② simultaneously: $3\times$① $-$ ②:

$$\begin{array}{r} \cancel{3x}+9y=21 \\ -\;\cancel{3x}-y=11 \\ \hline 10y=10 \end{array} \Rightarrow \therefore y=1$$

$y=1 \Rightarrow x=7-3y=7-3(1)=4 \quad \therefore x=4, y=1$

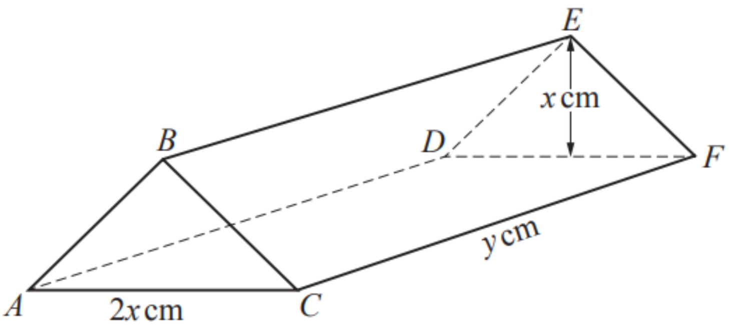
$\therefore (4,1)$ is the point of intersection between line & circle.

$r^2 = (x-x_1)+(y-y_1)$

Given $(4,1)$ & $(3,-2)$: $r^2 = (4-3)^2 + (1--2)^2 = 10$

$\boxed{\therefore \text{ Equation of Circle: } (x-3)^2+(y+2)^2 = 10}$

7



The diagram shows a model for the roof of a toy building. The roof is in the form of a solid triangular prism $ABCDEF$. The base $ACFD$ of the roof is a horizontal rectangle, and the cross-section ABC of the roof is an isosceles triangle with $AB = BC$.

The lengths of AC and CF are $2x$ cm and y cm respectively, and the height of BE above the base of the roof is x cm.

The total surface area of the **five** faces of the roof is 600 cm^2 and the volume of the roof is V cm^3.

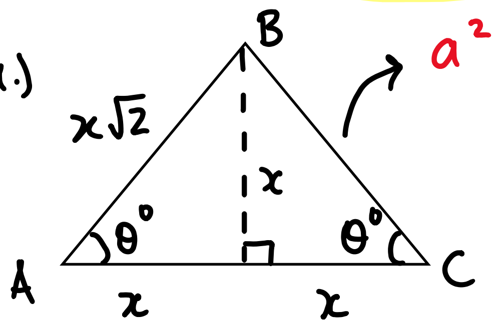
(a) Show that $V = kx(300 - x^2)$, where $k = \sqrt{a} + b$ and a and b are integers to be determined. **[6]**

(b) Use differentiation to determine the value of x for which the volume of the roof is a maximum. **[4]**

(c) Find the maximum volume of the roof. Give your answer in cm^3, correct to the nearest integer. **[1]**

(d) Explain why, for this roof, x must be less than a certain value, which you should state. **[2]**

(a.)



$a^2 + b^2 = c^2 \Rightarrow x^2 + x^2 = AB^2 = BC^2$

$\therefore AB = BC = \sqrt{2x^2} = x\sqrt{2}$

$$\text{Surface Area, } S = (2x)(y) + 2(x\sqrt{2})(y) + \cancel{2}\left(\tfrac{1}{\cancel{2}}\right)(2x)(x)$$

$$S = 2xy + 2\sqrt{2}\,xy + 2x^2 = 600$$

$$2xy(1+\sqrt{2}) + 2x^2 = 600$$

$$\therefore y = \frac{600 - 2x^2}{2x(1+\sqrt{2})} = \frac{300 - x^2}{x(1+\sqrt{2})}$$

$$V = \frac{1}{\cancel{2}}(\cancel{2}x)(x)(y) = x^2\left(\frac{300 - x^2}{x(1+\sqrt{2})}\right)$$

$$V = x(300 - x^2)\left(\frac{1}{1+\sqrt{2}}\right) = x(300 - x^2)(-1+\sqrt{2})$$

$$\therefore V = (\sqrt{2}-1)x(300 - x^2)$$

$\boxed{\therefore k = \sqrt{2} - 1, \text{ where } a = 2 \text{ \& } b = -1}$

(b.) $V = kx(300 - x^2) = 300kx - kx^3$

$$\frac{dV}{dx} = 300k - 3kx^2 = k(300 - 3x^2) = 0$$

$$300 - 3x^2 = 0$$

$$x = \pm\sqrt{\frac{300}{3}} = \pm\sqrt{100} = \pm 10$$

$$\frac{d^2V}{dx^2} = -6x$$

$x = 10 \Rightarrow \frac{d^2V}{dx^2} = -60 < 0 \therefore$ Maximum

$x = -10 \Rightarrow \frac{d^2V}{dx^2} = 60 > 0 \therefore$ Minimum

$\boxed{\therefore x = 10}$

(c.) $x = 10 \Rightarrow V = (\sqrt{2}-1)(10)(300 - 10^2) = 828.42\ldots$

$\boxed{\therefore V \approx 828 \text{ cm}^3 \text{ (3sf)}}$

(d.) V must be positive, so $300 - x^2 > 0$.

$\therefore x$ cannot exceed $\sqrt{300}$ cm. $x \leq \sqrt{300}$

Section B: Mechanics

Answer **all** the questions.

8 A particle is in equilibrium under the action of the following three forces:

$(2p\mathbf{i} - 4\mathbf{j})$ N, $(-3q\mathbf{i} + 5p\mathbf{j})$ N and $(-13\mathbf{i} - 6\mathbf{j})$ N.

Find the values of p and q. **[3]**

$\underline{i}$: $2p - 3q - 13 = 0$

j: $-4 + 5p - 6 = 0 \rightarrow \therefore p = \dfrac{4+6}{5} = 2$

$\therefore q = \dfrac{13 - 2(2)}{-3} = -3$

$\boxed{\therefore p = 2, q = -3}$

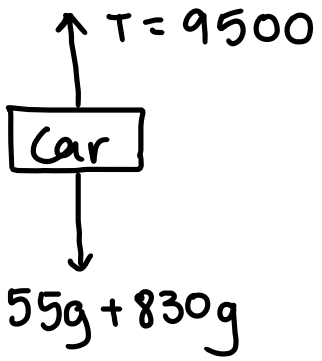
9 A crane lifts a car vertically. The car is inside a crate which is raised by the crane by means of a strong cable. The cable can withstand a maximum tension of 9500 N without breaking. The crate has a mass of 55 kg and the car has a mass of 830 kg.

(a) Find the maximum acceleration with which the crate and car can be raised. [2]

(b) Show on a clearly labelled diagram the forces acting on the **crate** while it is in motion. [1]

(c) Determine the magnitude of the reaction force between the crate and the car when they are ascending with maximum acceleration. [3]

(a.) $F = ma$



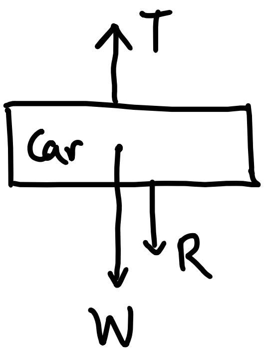
$g = 9.8$

$$9500 - 55g - 830g = (830 + 55)a$$

$$827 = 885a$$

$$\boxed{\therefore a = \frac{827}{885} = 0.9344\ldots \approx 0.934\ ms^{-2}\ (3sf)}$$

(b.)



(c.) $9500 - 55g - R = 55\left(\frac{827}{885}\right)$

$$8961 - R = \frac{9097}{177}$$

$$\boxed{\therefore R = 8961 - \frac{9097}{177} = 8909.6\ldots \approx 8910N\ (3sf)}$$

10 A particle P is moving in a straight line. At time t seconds P has velocity $v\,\mathrm{m\,s^{-1}}$ where $v=(2t+1)(3-t)$.

(a) Find the deceleration of P when $t=4$. **[2]**

(b) State the positive value of t for which P is instantaneously at rest. **[1]**

(c) Find the total distance that P travels between times $t=0$ and $t=4$. **[3]**

(a.) $v=(2t+1)(-t+3)=-2t^2+5t+3$

$a=\frac{dv}{dt}=-4t+5$

$t=4 \Rightarrow \therefore a=-4(4)+5=-11$

$\boxed{\therefore \text{Deceleration} = -11\ \mathrm{ms^{-2}}}$

(b.) $v=(2t+1)(-t+3)=0$

$2t+1=0 \qquad -t+3=0$

$t\neq-\frac{1}{2}<0 \qquad \therefore t=3>0$

$\boxed{\therefore t=3s}$

(c.) $d=\int v\,dt=\int -2t^2+5t+3\,dt=-\frac{2t^3}{3}+\frac{5t}{2}+3t+c$

$d=\int_0^4 v\,dt=\int_0^3 v\,dt+\int_3^4 v\,dt$

$=\left[-\frac{2}{3}t^3+\frac{5}{2}t+3t\right]_0^3+\left[-\frac{2}{3}t^3+\frac{5}{2}t+3t\right]_3^4$

$=\left(-\frac{2}{3}(3)^3+\frac{5}{2}(3)+3(3)\right)(2)-\left(-\frac{2}{3}(4)^3+\frac{5}{2}(4)+3(4)\right)$

$=-3-\left(-\frac{62}{3}\right)$

$=\frac{53}{3}$

$\boxed{\therefore \text{Total Distance of } P = 17.7\ \mathrm{m}\ (3sf)}$

11 A car starts from rest at a set of traffic lights and moves along a straight road with constant acceleration $4\,\mathrm{m\,s^{-2}}$. A motorcycle, travelling parallel to the car with constant speed $16\,\mathrm{m\,s^{-1}}$, passes the same traffic lights exactly 1.5 seconds after the car starts to move. The time after the car starts to move is denoted by t seconds.

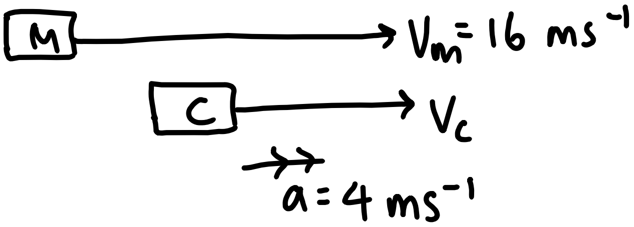
(a) Determine the two values of t at which the car and motorcycle are the same distance from the traffic lights. **[6]**

These two values of t are denoted by t_1 and t_2, where $t_1 < t_2$.

(b) Describe the relative positions of the car and the motorcycle when $t_1 < t < t_2$. **[1]**

(c) Determine the maximum distance between the car and the motorcycle when $t_1 < t < t_2$. **[3]**

(a)



Car: $s = \int\left(\int a\,dt\right)dt = \int 4t\,dt = \dfrac{4t^2}{2} = 2t^2$

Motorcycle: $s = vt = 16(t - 1.5)$

$2t^2 = 16(t-1.5)$

$2t^2 = 16t - 24$

$2t^2 - 16t + 24 = 0$

$\div 2 \qquad \div 2$

$t^2 - 8t + 12 = 0$

$(t-6)(t-2) = 0$

$t = 6 \qquad t = 2$

$\boxed{\therefore t_1 = 2\mathrm{s},\ t_2 = 6\mathrm{s}}$

(b.) $t_1 < t < t_2$

Motorcycle is ahead of the car.

(c.) Maximum distance is at $t = 4$.

$f(t) = -(2t^2 - 16t + 24) = -2t^2 + 16t - 24$

$f(4) = -2(4)^2 + 16(4) - 24 = 8\text{ m}$

$\boxed{\therefore \text{Maximum distance} = 8\text{ m}}$