Oxford Cambridge and RSA

AS Level Mathematics A **Model Solutions**

H230/02 Pure Mathematics and Mechanics

Wednesday 23 May 2018 – Morning

Time allowed: 1 hour 30 minutes

You must have:
- Printed Answer Booklet

You may use:
- a scientific or graphical calculator

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INSTRUCTIONS
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Complete the boxes provided on the Printed Answer Booklet with your name, centre number and candidate number.
- Answer **all** the questions.
- **Write your answer to each question in the space provided in the Printed Answer Booklet.** Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- Do **not** write in the barcodes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The acceleration due to gravity is denoted by $g\,\mathrm{m\,s^{-2}}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

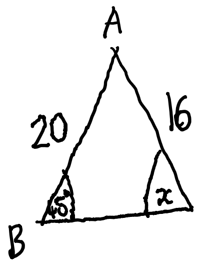
INFORMATION
- The total number of marks for this paper is **75**.
- The marks for each question are shown in brackets **[ ]**.
- **You are reminded of the need for clear presentation in your answers.**
- The Printed Answer Booklet consists of **12** pages. The Question Paper consists of **8** pages.

1 In triangle ABC, $AB = 20\,\text{cm}$ and angle $B = 45°$.

(i) Given that $AC = 16\,\text{cm}$, find the two possible values for angle C, correct to 1 decimal place. **[4]**

(ii) Given instead that the area of the triangle is $75\sqrt{2}\,\text{cm}^2$, find BC. **[2]**

1) i)



Finding x using sin rule $\quad \dfrac{\sin A}{A} = \dfrac{\sin B}{b}$

$$\frac{\sin x}{20} = \frac{\sin 45}{16}$$

$$\Rightarrow \sin x = \frac{20 \sin 45}{16}$$

$$\Rightarrow \sin x = \frac{5\sqrt{2}}{8}$$

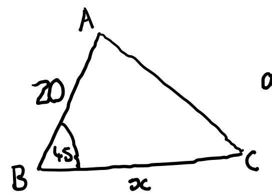
taking $\sin^{-1}\left(\dfrac{5\sqrt{2}}{8}\right)$ we get

$x = 62.1$

also $\quad 180 - 62.1 = 117.9$

so $\quad x = 62.1°$ or $117.9°$

ii)



area $= 75\sqrt{2}$

using area $= \frac{1}{2}ab\sin C$

$75\sqrt{2} = \frac{1}{2} \cdot 20 \cdot x \cdot \sin 45$

$\Rightarrow 75\sqrt{2} = \frac{20x}{2}\sin 45$

$\Rightarrow 75\sqrt{2} = \frac{10\sqrt{2}}{2}x$

$\Rightarrow \frac{150\sqrt{2}}{10\sqrt{2}} = x$

$\Rightarrow x = 15$

2 **(i)** The curve $y = \frac{2}{3+x}$ is translated by four units in the positive x-direction. State the equation of the curve after it has been translated. **[2]**

(ii) Describe fully the single transformation that transforms the curve $y = \frac{2}{3+x}$ to $y = \frac{5}{3+x}$. **[2]**

2) i) $y = \frac{2}{3+x}$

translation is $f(x-4)$

so $y = \frac{2}{3+(x-4)}$

$\Rightarrow y = \frac{2}{x-1}$

ii) $y = \frac{2}{3+x} \longrightarrow y = \frac{5}{3+x}$

$\frac{5}{3+x} = \frac{2}{3+x} \cdot \frac{5}{2}$

so it is a stretch of $\frac{5}{2}$ parallel to y

3 In each of the following cases choose one of the statements

$$P \Rightarrow Q \qquad P \Leftarrow Q \qquad P \Leftrightarrow Q$$

to describe the relationship between P and Q.

(i) P: $y = 3x^5 - 4x^2 + 12x$
Q: $\frac{dy}{dx} = 15x^4 - 8x + 12$ **[1]**

(ii) P: $x^5 - 32 = 0$ where x is real
Q: $x = 2$ **[1]**

(iii) P: $\ln y < 0$
Q: $y < 1$ **[1]**

3)i) $\frac{d}{dx}\left(3x^5 - 4x^2 + 12x\right) = 15x^4 - 8x + 12$

however $\int 15x^4 - 8x + 12 \, dx = 3x^5 - 4x^2 + 12x + C$

so $P \Rightarrow Q$

ii) $x^5 - 32 = 0$

$\Rightarrow x^5 = 32$

$\Rightarrow x = 2$ so $P \Rightarrow Q$

Substituting $x = 2$ into $x^5 - 32$

$32 - 32 = 0$ so $Q \Rightarrow P$

Therefore $P \Leftrightarrow Q$

iii) $\ln y < 0$

$\Rightarrow y < e^0$

$\Rightarrow y < 1$ so $P \Rightarrow Q$

take $y < 1$ for example $y = -3$

$\ln -3$ does not exist

so $Q \not\Rightarrow P$

Therefore $P \Rightarrow Q$

4 **(i)** Express $4x^2 - 12x + 11$ in the form $a(x+b)^2 + c$. **[3]**

(ii) State the number of real roots of the equation $4x^2 - 12x + 11 = 0$. **[1]**

(iii) Explain fully how the value of r is related to the number of real roots of the equation $p(x+q)^2 + r = 0$ where p, q and r are real constants and $p > 0$. **[2]**

4) i) $4x^2 - 12x + 11$

$= 4\left(x^2 - 3x + \frac{11}{4}\right)$

$= 4\left[\left(x - \frac{3}{2}\right)^2 - \frac{9}{4} + \frac{11}{4}\right]$

$= 4\left[\left(x - \frac{3}{2}\right)^2 + \frac{2}{4}\right]$

$= 4\left(x - \frac{3}{2}\right)^2 + 2$

$a = 4 \quad b = -\frac{3}{2} \quad c = 2$

ii) $4x^2 - 12x + 11 = 0$

discriminant : $b^2 - 4ac$

$(-12)^2 - 4 \cdot 4 \cdot 11$

$= 144 - 176 = -32$

so discriminant < 0 so no real roots

iii) As r is the turning point

$r = 0 \Rightarrow 1$ real root

$r < 0 \Rightarrow 2$ real roots

$r > 0 \Rightarrow$ no real roots

5 In this question you must show detailed reasoning.

The line $x + 5y = k$ is a tangent to the curve $x^2 - 4y = 10$. Find the value of the constant k. **[5]**

5) $x + 5y = k \Rightarrow x = k - 5y$ ①

$x^2 - 4y = 10$ ②

substituting ① into ② we get

$(k - 5y)^2 - 4y = 10$

$\Rightarrow k^2 - 10ky + 25y^2 - 4y = 10$

$\Rightarrow 25y^2 - 10ky - 4y + k^2 - 10 = 0$

Grouping k terms we get

$25y^2 + (-10k - 4)y + (k^2 - 10) = 0$

Tangent $\Rightarrow$ discriminant $= 0$

so $b^2 - 4ac = (-10k-4)^2 - 4 \cdot 25 \cdot (k^2-10) = 0$

$\Rightarrow 100k^2 + 80k + 16 - 100(k^2-10) = 0$

$\Rightarrow 100k^2 + 80k + 16 - 100k^2 + 1000 = 0$

$\Rightarrow 80k + 1016 = 0$

$\Rightarrow k = -\frac{127}{10}$

6 A pan of water is heated until it reaches 100 °C. Once the water reaches 100 °C, the heat is switched off and the temperature T °C of the water decreases. The temperature of the water is modelled by the equation

$$T = 25 + ae^{-kt},$$

where t denotes the time, in minutes, after the heat is switched off and a and k are positive constants.

(i) Write down the value of a. **[1]**

(ii) Explain what the value of 25 represents in the equation $T = 25 + ae^{-kt}$. **[1]**

When the heat is switched off, the initial rate of decrease of the temperature of the water is 15 °C per minute.

(iii) Calculate the value of k. **[3]**

(iv) Find the time taken for the temperature of the water to drop from 100 °C to 45 °C. **[3]**

(v) A second pan of water is heated, but the heat is turned off when the water is at a temperature of less than 100 °C. Suggest how the equation for the temperature as the water cools would be modified by this. **[1]**

6) i) $T = 25 + ae^{-kt}$

at time $t = 0$ $T = 100$

so substituting:

$100 = 25 + ae^{0}$

$\Rightarrow 100 = 25 + a$

$\Rightarrow a = 75$

ii) As t gets bigger $ae^{-kt} \to 0$

so 25 is the lowest temperature

iii) $T = 25 + 75e^{-kt}$

$$\frac{dT}{dt} = -75ke^{-kt}$$

at $t = 0$ $\frac{dT}{dt} = -15$

so

$$-15 = -75ke^{0}$$

$\Rightarrow -15 = -75k$

$\Rightarrow k = \frac{1}{5}$

iv) $45 = 25 + 75e^{-\frac{1}{5}t}$

$$\frac{20}{75} = e^{-\frac{1}{5}t}$$

$$\ln\left(\frac{20}{75}\right) = -\frac{1}{5}t$$

$\Rightarrow t = 6.6$ mins

v) By decreasing the value of a

7 **(i)** Show that the equation

$$2\sin x\tan x = \cos x + 5$$

can be expressed in the form

$$3\cos^2 x + 5\cos x - 2 = 0.$$ **[3]**

(ii) Hence solve the equation

$$2\sin 2\theta \tan 2\theta = \cos 2\theta + 5,$$

giving all values of θ between 0° and 180°, correct to 1 decimal place. **[5]**

7) i) $2\sin x \tan x = \cos x + 5$

$2\sin x \cdot \frac{\sin x}{\cos x} = \cos x + 5$

$\frac{2\sin^2 x}{\cos x} = \cos x + 5$

$2\sin^2 x = \cos^2 x + 5\cos x$

$2(1-\cos^2 x) = \cos^2 x + 5\cos x$

$2 - 2\cos^2 x = \cos^2 x + 5\cos x$

$\Rightarrow 3\cos^2 x + 5\cos x - 2 = 0$

ii) replacing x with 2θ in the previous part

$3\cos^2 2\theta + 5\cos 2\theta - 2 = 0$

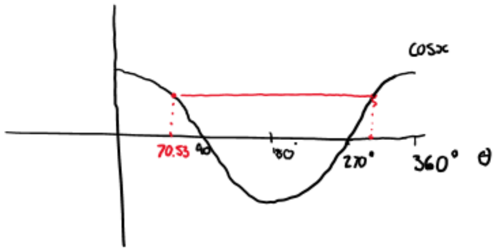
$\Rightarrow (3\cos 2\theta - 1)(\cos 2\theta + 2) = 0$

so $\underline{3\cos 2\theta - 1 = 0}$ and $\underline{\cos 2\theta + 2 = 0}$ does not work so we reject

$\Rightarrow 3\cos 2\theta = 1$

$\Rightarrow \cos 2\theta = \frac{1}{3}$

$\cos^{-1}\left(\frac{1}{3}\right) = 70.53$



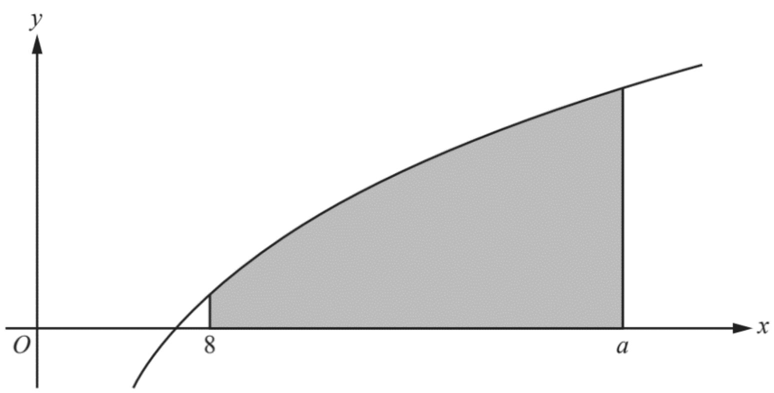
so $2\theta = 70.53$

$2\theta = 360 - 70.53 = 289.47$

so $\theta = 35.3$ $\qquad \theta = 144.7$

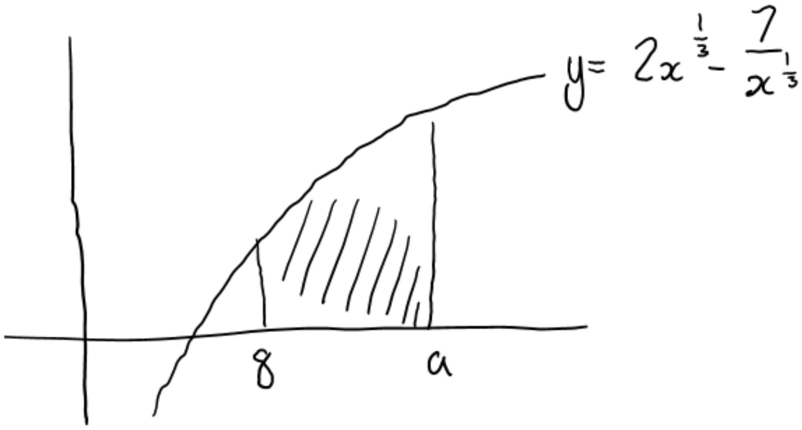
8 In this question you must show detailed reasoning.

The diagram shows part of the graph of $y = 2x^{\frac{1}{3}} - \frac{7}{x^{\frac{1}{3}}}$. The shaded region is enclosed by the curve, the x-axis and the lines $x = 8$ and $x = a$, where $a > 8$.



Given that the area of the shaded region is 45 square units, find the value of a. **[9]**

8)



area = 45

$$\int_8^a 2x^{\frac{1}{3}} - \frac{7}{x^{\frac{1}{3}}}\,dx = 45$$

$$\Rightarrow \int_8^a 2x^{\frac{1}{3}} - 7x^{-\frac{1}{3}}\,dx = 45$$

$$\Rightarrow \left[2\cdot\frac{3}{4}x^{\frac{4}{3}} - 7\cdot\frac{3}{2}x^{\frac{2}{3}}\right]_8^a = 45$$

$$\Rightarrow \left[\frac{3}{2}x^{\frac{4}{3}} - \frac{21}{2}x^{\frac{2}{3}}\right]_8^a = 45$$

$$\Rightarrow \frac{3}{2}a^{\frac{4}{3}} - \frac{21}{2}a^{\frac{2}{3}} - \left(\frac{3}{2}\cdot 8^{\frac{4}{3}} - \frac{21}{2}\cdot 8^{\frac{2}{3}}\right) = 45$$

$$\Rightarrow \frac{3}{2}a^{\frac{4}{3}} - \frac{21}{2}a^{\frac{2}{3}} - (24 - 42) = 45$$

$$\Rightarrow \frac{3}{2}a^{\frac{4}{3}} - \frac{21}{2}a^{\frac{2}{3}} = 27$$

$$\Rightarrow \quad 3a^{\frac{4}{3}} - 21a^{\frac{2}{3}} = 54$$

$$a^{\frac{4}{3}} - 7a^{\frac{2}{3}} = 18$$

$$\Rightarrow \quad a^{\frac{4}{3}} - 7a^{\frac{2}{3}} - 18 = 0$$

Set $X = a^{\frac{2}{3}}$

the equation becomes

$$X^2 - 7X - 18 = 0$$

$$(X-9)(X+2) = 0$$

so $X = 9$ or -2

so $a^{\frac{2}{3}} = 9$ or $a^{\frac{2}{3}} = -2$

$a^{\frac{2}{3}} = 9$

$\Rightarrow a = 27$

$a^{\frac{2}{3}} = -2$

does not work

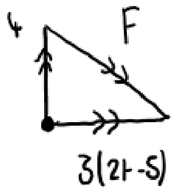
so $a = 27$

9 In this question the horizontal unit vectors **i** and **j** are in the directions east and north respectively.

A model ship of mass 2 kg is moving so that its acceleration vector $\mathbf{a}\,\mathrm{m\,s^{-2}}$ at time t seconds is given by $\mathbf{a} = 3(2t-5)\mathbf{i}+4\mathbf{j}$. When $t = T$, the magnitude of the horizontal force acting on the ship is 10 N.

Find the possible values of T. **[4]**

9) $a = 3(2t-5)i + 4j$



$F = \sqrt{(ma)^2}$

$F = \sqrt{2^2(3^2(2T-5)^2) + 2^2(4^2)}$

$F = \sqrt{36(2T-5)^2 + 64}$

$F = 10$

so

$10 = \sqrt{36(2T-5)^2 + 64}$

$100 = 36(2T-5)^2 + 64$

$36 = 36(2T-5)^2$

$1 = (2T-5)^2$

so $2T-5 = \pm 1$

$\underline{+1}$

$2T-5 = 1$

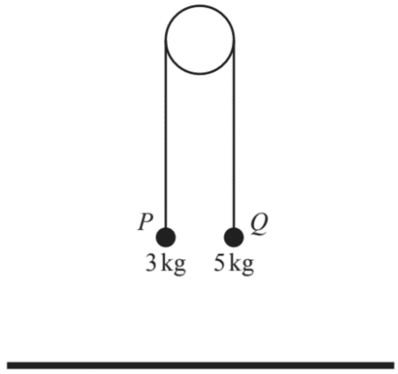
$\Rightarrow T = 3$

$\underline{-1}$

$2T-5 = -1$

$\Rightarrow T = 2$

10 Particles P and Q, of masses 3 kg and 5 kg respectively, are attached to the ends of a light inextensible string. The string passes over a smooth fixed pulley. The system is held at rest with the string taut. The hanging parts of the string are vertical and P and Q are above a horizontal plane (see diagram).

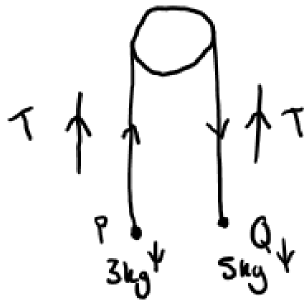


(i) Find the tension in the string immediately after the particles are released. **[4]**

After descending 2.5 m, Q strikes the plane and is immediately brought to rest. It is given that P does not reach the pulley in the subsequent motion.

(ii) Find the distance travelled by P between the instant when Q strikes the plane and the instant when the string becomes taut again. **[4]**

10)



Using F = ma on both sides of the pulley

Left side

tension = T

$T - 3g = 3a$

Right side

$5g - T = 5a$

Solving simultanously for T ie. removing a

$$\begin{cases} T - 3g = 3a \Rightarrow \dfrac{T-3g}{3} = a \\ 5g - T = 5a \Rightarrow \dfrac{5g-T}{5} = a \end{cases}$$

$$\frac{T-3g}{3} = \frac{5g-T}{5}$$

$\Rightarrow \quad 5(T-3g) = 3(5g-T)$

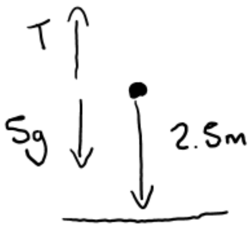
$\Rightarrow \quad 5T - 15g = 15g - 3T$

$\Rightarrow \quad 8T = 30g$

$T = \frac{30g}{8} \qquad g = 9.8$

so $\quad T = 36.75$

ii) Q



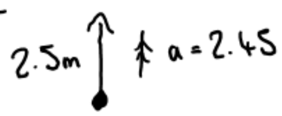
Using F = ma

$(5g - T) = 5a$

$T = 36.75 \quad \Rightarrow \quad 12.25 = 5a$

$\Rightarrow \quad 2.45 = a$

P



when taunt again

$v = 0$

$S = 2.5$
$U = 0$
$V = $ Find
$A = 2.45$
$T =$

so using $v^2 = u^2 + 2as$ to find v

$v^2 = 0 + 2 \cdot 2.45 \cdot 2.5$

$\Rightarrow v^2 = 12.25$
$v = \sqrt{12.25}$

Now Q moves freely so

S Find
$U = \sqrt{12.25}$
$V = 0$
$A = -g$
T

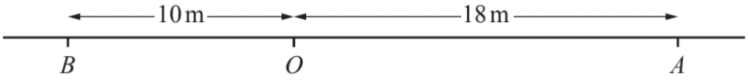
$v^2 = u^2 + 2as$

$\Rightarrow 0 = 12.25 + 2 \cdot -g \cdot s$

$-12.25 = -2gs$

$\Rightarrow \frac{12.25}{2g} = s \quad \Rightarrow s = 0.625m$

11



A particle P is moving along a straight line with constant acceleration. Initially the particle is at O. After 9 s, P is at a point A, where $OA = 18$ m (see diagram) and the velocity of P at A is $8\,\mathrm{m\,s^{-1}}$ in the direction $\overrightarrow{OA}$.

(i) **(a)** Show that the initial speed of P is $4\,\mathrm{m\,s^{-1}}$. **[2]**

(b) Find the acceleration of P. **[2]**

B is a point on the line such that $OB = 10$ m, as shown in the diagram.

(ii) Show that P is never at point B. **[4]**

A second particle Q moves along the same straight line, but has variable acceleration. Initially Q is at O, and the displacement of Q from O at time t seconds is given by

$$x = at^3 + bt^2 + ct,$$

where a, b and c are constants.

It is given that
- the velocity and acceleration of Q at the point O are the same as those of P at O,
- Q reaches the point A when $t = 6$.

(iii) Find the velocity of Q at A. **[5]**

11) i a)

$S = 18$ so using $S = \left(\frac{u+v}{2}\right)t$

$U = $ Find

$V = 8$ $\quad 18 = \left(\frac{u+8}{2}\right)9 \Rightarrow 2 = \frac{u+8}{2}$

$A =$

$T = 9$ $\quad \Rightarrow 4 = u+8 \Rightarrow u = -4$ so the speed of p is $4\,\mathrm{ms^{-1}}$

i b)

$S = 18$
$U =$
$V = 8$
$A =$ Find
$T = 9$

$$S = vt - \frac{1}{2}at^2$$

$$18 = 72 - \frac{1}{2} \cdot a \cdot 81$$

$$-54 = -\frac{81}{2}a$$

$$\Rightarrow a = \frac{4}{3}\ ms^{-2}$$

ii) Finding when the particle stops moving in the OB direction

$S =$
$U = -4$
$V = 0$
$A = \frac{4}{3}$
$T =$ Find

$$v = u + at$$

$$0 = -4 + \frac{4}{3}t$$

$$4 = \frac{4}{3}t \Rightarrow t = 3$$

so after 3 seconds we find the displacement

$S =$ Find
$U = -4$
$V =$
$A = \frac{4}{3}$
$T = 3$

$$S = ut + \frac{1}{2}at^2$$

$$-s_{max} = -4 \cdot 3 + \frac{1}{2}\left(\frac{4}{3}\right) \cdot 3^2$$

$\Rightarrow S_{max} = 6 < 10m$ so particle never reaches B

iii) need to find a, b, c

velocity = $\frac{dx}{dt}$ acceleration = $\frac{d^2x}{dt^2}$

$x = at^3 + bt^2 + ct$

$\frac{dx}{dt} = 3at^2 + 2bt + c$ = velocity

$\frac{d^2x}{dt^2} = 6at + 2b$ - acceleration

acceleration at $t = 0$ is $\frac{4}{3}$

$\frac{4}{3} = 6a \cdot 0 + 2b$

$\Rightarrow \frac{4}{3} = 2b \Rightarrow b = \frac{2}{3}$

velocity at $t = 0$ is -4

$-4 = 3a \cdot 0 + 2b \cdot 0 + c$

so $c = -4$

displacement at $t = 6$ is 18

$x = at^3 + bt^2 + ct$ $b = \frac{2}{3}$ $c = -4$

$18 = a \cdot 6^3 + b \cdot 6^2 + 6c$

$18 = 216a + 36b + 6c$

$18 = 216a + 36 \cdot \frac{2}{3} + 6 \cdot -4$

$18 = 216a$

$\Rightarrow a = \frac{1}{12}$

Velocity at $t = 6$ (when Q is at A)

$a = \frac{1}{12}$ $b = \frac{2}{3}$ $c = -4$

$\frac{dx}{dt} = 3at^2 + 2bt + c$

$\frac{1}{4} \cdot 6^2 + \frac{4}{3} \cdot 6 - 4 = 13 \text{ ms}^{-1}$