Oxford Cambridge and RSA

Friday 14 June 2019 – Afternoon

A Level Mathematics B (MEI)

H640/03 Pure Mathematics and Comprehension

Time allowed: 2 hours

You must have:
- Printed Answer Booklet
- Insert

You may use:
- a scientific or graphical calculator

INSTRUCTIONS
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Answer **all** the questions.
- **Write your answer to each question in the space provided in the Printed Answer Booklet.** If additional space is required, you should use the lined page(s) at the end of the Printed Answer Booklet. The question number(s) must be clearly shown.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION
- The total mark for this paper is **75**.
- The marks for each question are shown in brackets **[ ]**.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is used. You should communicate your method with correct reasoning.
- The Printed Answer Booklet consists of **16** pages. The Question Paper consists of **8** pages.

Formulae A Level Mathematics B (MEI) (H640)

Arithmetic series

$$S_n = \tfrac{1}{2}n(a+l) = \tfrac{1}{2}n\{2a+(n-1)d\}$$

Geometric series

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_\infty = \frac{a}{1-r} \text{ for } |r| < 1$$

Binomial series

$$(a+b)^n = a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \ldots + {}^nC_r a^{n-r}b^r + \ldots + b^n \qquad (n \in \mathbb{N}),$$

where $${}^nC_r = {}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots \qquad (|x| < 1,\ n \in \mathbb{R})$$

Differentiation

$f(x)$	$f'(x)$
$\tan kx$	$k\sec^2 kx$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$

Quotient Rule $y = \frac{u}{v}$, $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Differentiation from first principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Integration

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

$$\int f'(x)\big(f(x)\big)^n\,dx = \frac{1}{n+1}\big(f(x)\big)^{n+1} + c$$

Integration by parts $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$

Small angle approximations

$\sin\theta \approx \theta$, $\cos\theta \approx 1 - \frac{1}{2}\theta^2$, $\tan\theta \approx \theta$ where θ is measured in radians

Trigonometric identities

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \qquad \left(A \pm B \neq (k + \tfrac{1}{2})\pi\right)$$

Numerical methods

Trapezium rule: $\int_a^b y\,\mathrm{d}x \approx \frac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\}$, where $h = \frac{b-a}{n}$

The Newton-Raphson iteration for solving $\mathrm{f}(x) = 0$: $x_{n+1} = x_n - \frac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)}$

Probability

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$$

$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B \mid A) = \mathrm{P}(B)\mathrm{P}(A \mid B)$ **or** $\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$

Sample variance

$$s^2 = \frac{1}{n-1}S_{xx} \text{ where } S_{xx} = \sum(x_i - \bar{x})^2 = \sum x_i^2 - \frac{(\sum x_i)^2}{n} = \sum x_i^2 - n\bar{x}^2$$

Standard deviation, $s = \sqrt{\text{variance}}$

The binomial distribution

If $X \sim \mathrm{B}(n, p)$ then $\mathrm{P}(X = r) = {}^n\mathrm{C}_r\, p^r q^{n-r}$ where $q = 1 - p$

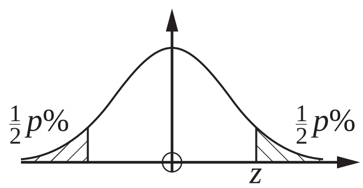
Mean of X is np

Hypothesis testing for the mean of a Normal distribution

If $X \sim \mathrm{N}(\mu, \sigma^2)$ then $\bar{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$ and $\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim \mathrm{N}(0,1)$

Percentage points of the Normal distribution

p	10	5	2	1
z	1.645	1.960	2.326	2.576



Kinematics

Motion in a straight line

$v = u + at$

$s = ut + \frac{1}{2}at^2$

$s = \frac{1}{2}(u + v)t$

$v^2 = u^2 + 2as$

$s = vt - \frac{1}{2}at^2$

Motion in two dimensions

$\mathbf{v} = \mathbf{u} + \mathbf{a}t$

$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$

$\mathbf{s} = \frac{1}{2}(\mathbf{u} + \mathbf{v})t$

$\mathbf{s} = \mathbf{v}t - \frac{1}{2}\mathbf{a}t^2$

Answer **all** the questions.

Section A (60 marks)

1 The function $f(x)$ is defined for all real x by

$f(x) = 3x - 2$.

(a) Find an expression for $f^{-1}(x)$. **[2]**

(b) Sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same diagram. **[2]**

(c) Find the set of values of x for which $f(x) > f^{-1}(x)$. **[2]**

2 **(a)** Find the transformation which maps the curve $y = x^2$ to the curve $y = x^2 + 8x - 7$. **[4]**

(b) Write down the coordinates of the turning point of $y = x^2 + 8x - 7$. **[1]**

3 **(a)** Express $\dfrac{1}{(x+2)(x+3)}$ in partial fractions. **[3]**

(b) Find $\displaystyle\int \frac{1}{(x+2)(x+3)}\,dx$ in the form $\ln(f(x)) + c$, where c is the constant of integration and $f(x)$ is a function to be determined. **[3]**

4 **In this question you must show detailed reasoning.**

Show that $\dfrac{1}{\sqrt{10}+\sqrt{11}} + \dfrac{1}{\sqrt{11}+\sqrt{12}} + \dfrac{1}{\sqrt{12}+\sqrt{13}} = \dfrac{3}{\sqrt{10}+\sqrt{13}}$. **[3]**

5 A student's attempt to prove by contradiction that there is no largest prime number is shown below.

> If there is a largest prime, list all the primes.
>
> Multiply all the primes and add 1.
>
> The new number is not divisible by any of the primes in the list and so it must be a new prime.

The proof is incorrect and incomplete.
Write a correct version of the proof. **[3]**

6 A circle has centre C (10, 4). The *x*-axis is a tangent to the circle, as shown in Fig. 6.

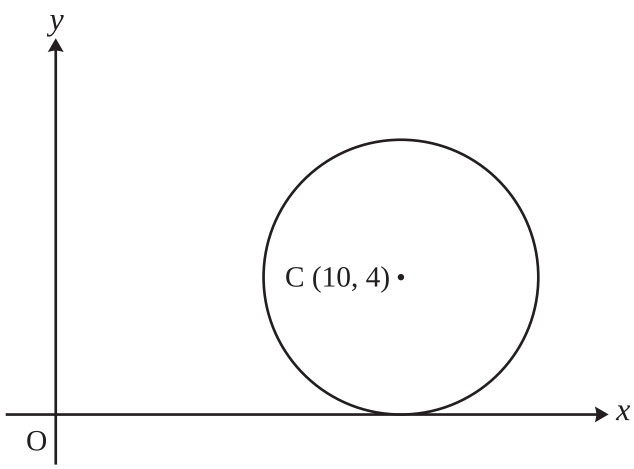


Fig. 6

(a) Find the equation of the circle. **[2]**

(b) Show that the line $y = x$ is not a tangent to the circle. **[4]**

(c) Write down the position vector of the midpoint of OC. **[1]**

7 **In this question you must show detailed reasoning.**

(a) Express $\ln 3 \times \ln 9 \times \ln 27$ in terms of $\ln 3$. **[2]**

(b) Hence show that $\ln 3 \times \ln 9 \times \ln 27 > 6$. **[2]**

8 In this question you must show detailed reasoning.

A is the point (1, 0), B is the point (1, 1) and D is the point where the tangent to the curve $y = x^3$ at B crosses the x-axis, as shown in Fig. 8. The tangent meets the y-axis at E.

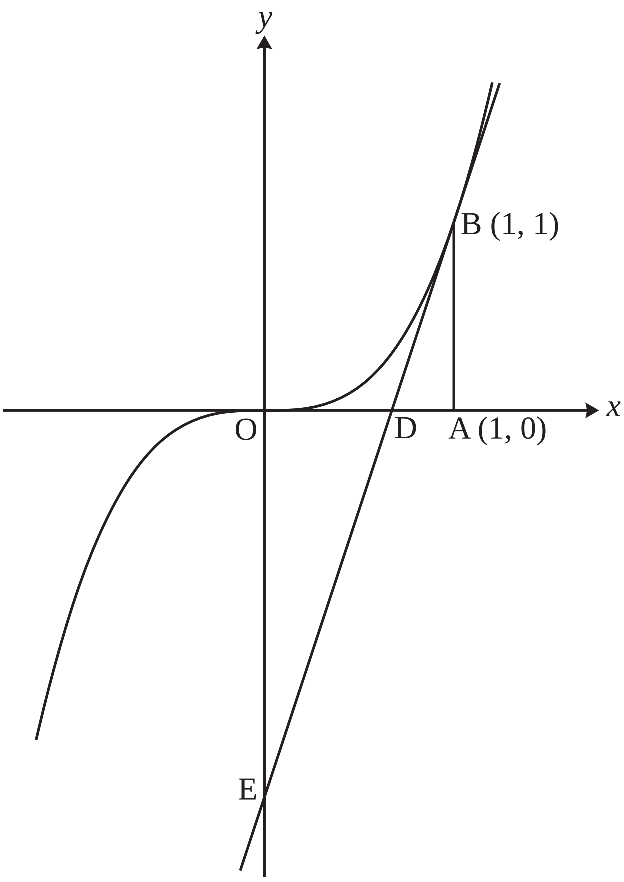


Fig. 8

(a) Find the area of triangle ODE. **[6]**

(b) Find the area of the region bounded by the curve $y = x^3$, the tangent at B and the y-axis. **[4]**

9 **In this question you must show detailed reasoning.**

The curve $xy + y^2 = 8$ is shown in Fig. 9.

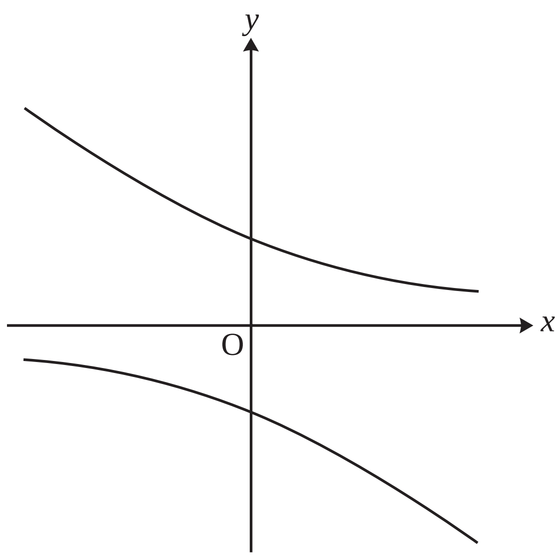


Fig. 9

Find the coordinates of the points on the curve at which the normal has gradient 2. **[6]**

10 Show that $f(x) = \dfrac{e^x}{1 + e^x}$ is an increasing function for all values of x. **[4]**

11 By using the substitution $u = 1 + \sqrt{x}$, find $\displaystyle\int \frac{x}{1 + \sqrt{x}}\,dx$. **[6]**

Turn over

Answer **all** the questions.

Section B (15 marks)

The questions in this section refer to the article on the Insert. You should read the article before attempting the questions.

12 Show that the equation of the line in Fig. C2 is $ry + hx = hr$, as given in line 24. **[2]**

13 (a) (i) Show that the cross-sectional area in Fig. C3.2 is $\pi x(2r - x)$. **[2]**

(ii) Hence show that the cross-sectional area is $\frac{\pi r^2}{h^2}(h^2 - y^2)$, as given in line 37. **[2]**

(b) Verify that the formula $\frac{\pi r^2}{h^2}(h^2 - y^2)$ for the cross-sectional area is also valid for

(i) Fig. C3.1, **[1]**

(ii) Fig. C3.3. **[1]**

14 (a) Express $\lim_{\delta y \to 0} \sum_{0}^{h} (h^2 - y^2)\delta y$ as an integral. **[1]**

(b) Hence show that $V = \frac{2}{3}\pi r^2 h$, as given in line 41. **[3]**

15 A typical tube of toothpaste measures 5.4 cm across the straight edge at the top and is 12 cm high. It contains 75 ml of toothpaste so it needs to have an internal volume of $75\,\text{cm}^3$.

Comment on the accuracy of the formula $V = \frac{2}{3}\pi r^2 h$, as given in line 41, for the volume in this case. **[3]**

END OF QUESTION PAPER