OCR
Oxford Cambridge and RSA

Friday 14 June 2019 – Afternoon

A Level Mathematics B (MEI)

H640/03 Pure Mathematics and Comprehension

Time allowed: 2 hours

You must have:
- Printed Answer Booklet
- Insert

You may use:
- a scientific or graphical calculator

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INSTRUCTIONS

- Use black ink. HB pencil may be used for graphs and diagrams only.
- Answer **all** the questions.
- **Write your answer to each question in the space provided in the Printed Answer Booklet.** If additional space is required, you should use the lined page(s) at the end of the Printed Answer Booklet. The question number(s) must be clearly shown.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION

- The total mark for this paper is **75**.
- The marks for each question are shown in brackets **[ ]**.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is used. You should communicate your method with correct reasoning.
- The Printed Answer Booklet consists of **16** pages. The Question Paper consists of **8** pages.

Formulae A Level Mathematics B (MEI) (H640)

Arithmetic series

$$S_n = \tfrac{1}{2}n(a+l) = \tfrac{1}{2}n\{2a+(n-1)d\}$$

Geometric series

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_\infty = \frac{a}{1-r} \text{ for } |r| < 1$$

Binomial series

$$(a+b)^n = a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \ldots + {}^nC_r a^{n-r}b^r + \ldots + b^n \qquad (n \in \mathbb{N}),$$

where $${}^nC_r = {}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots \qquad (|x| < 1,\ n \in \mathbb{R})$$

Differentiation

$f(x)$	$f'(x)$
$\tan kx$	$k\sec^2 kx$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$

Quotient Rule $y = \dfrac{u}{v}$, $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

Differentiation from first principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Integration

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

$$\int f'(x)(f(x))^n\,dx = \frac{1}{n+1}(f(x))^{n+1} + c$$

Integration by parts $\displaystyle\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$

Small angle approximations

$\sin\theta \approx \theta$, $\cos\theta \approx 1 - \tfrac{1}{2}\theta^2$, $\tan\theta \approx \theta$ where θ is measured in radians

Trigonometric identities

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \qquad \left(A \pm B \neq (k+\tfrac{1}{2})\pi\right)$

Numerical methods

Trapezium rule: $\int_a^b y\,\mathrm{d}x \approx \frac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\}$, where $h = \dfrac{b-a}{n}$

The Newton-Raphson iteration for solving $\mathrm{f}(x) = 0$: $x_{n+1} = x_n - \dfrac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)}$

Probability

$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B \mid A) = \mathrm{P}(B)\mathrm{P}(A \mid B)$ **or** $\mathrm{P}(A \mid B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$

Sample variance

$s^2 = \dfrac{1}{n-1}S_{xx}$ where $S_{xx} = \Sigma(x_i - \bar{x})^2 = \Sigma x_i^2 - \dfrac{(\Sigma x_i)^2}{n} = \Sigma x_i^2 - n\bar{x}^2$

Standard deviation, $s = \sqrt{\text{variance}}$

The binomial distribution

If $X \sim \mathrm{B}(n, p)$ then $\mathrm{P}(X = r) = {}^n\mathrm{C}_r p^r q^{n-r}$ where $q = 1 - p$

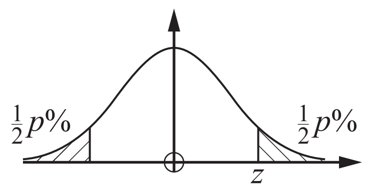
Mean of X is np

Hypothesis testing for the mean of a Normal distribution

If $X \sim \mathrm{N}(\mu, \sigma^2)$ then $\overline{X} \sim \mathrm{N}\left(\mu, \dfrac{\sigma^2}{n}\right)$ and $\dfrac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim \mathrm{N}(0, 1)$

Percentage points of the Normal distribution

p	10	5	2	1
z	1.645	1.960	2.326	2.576



Kinematics

Motion in a straight line

$v = u + at$

$s = ut + \frac{1}{2}at^2$

$s = \frac{1}{2}(u + v)t$

$v^2 = u^2 + 2as$

$s = vt - \frac{1}{2}at^2$

Motion in two dimensions

$\mathbf{v} = \mathbf{u} + \mathbf{a}t$

$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$

$\mathbf{s} = \frac{1}{2}(\mathbf{u} + \mathbf{v})t$

$\mathbf{s} = \mathbf{v}t - \frac{1}{2}\mathbf{a}t^2$

Answer **all** the questions.

Section A (60 marks)

1 The function $\mathrm{f}(x)$ is defined for all real x by

$\mathrm{f}(x) = 3x - 2.$

(a) Find an expression for $\mathrm{f}^{-1}(x)$. **[2]**

(b) Sketch the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$ on the same diagram. **[2]**

(c) Find the set of values of x for which $\mathrm{f}(x) > \mathrm{f}^{-1}(x)$. **[2]**

a) $f(x) = 3x - 2$

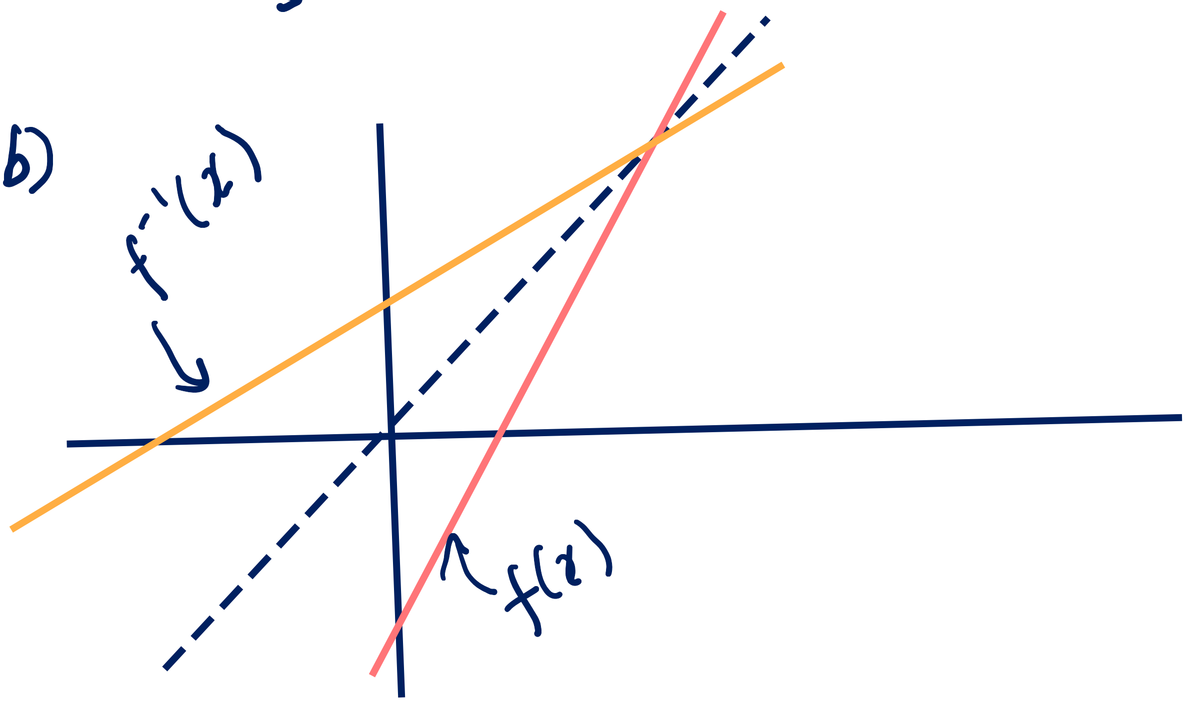
$y = 3x - 2$

$y + 2 = 3x$

$\frac{y+2}{3} = x$

$\therefore f^{-1}(x) = \frac{x+2}{3}$

b)



c) $f(x) > f^{-1}(x)$

$$3 \times \quad 3x-2 > \frac{x+2}{3} \quad \times 3$$

$$9x-6 > x+2$$

$$9x-x > 2+6$$

$$\frac{8x}{8} > \frac{8}{8}$$

$$x > 1$$

2 **(a)** Find the transformation which maps the curve $y = x^2$ to the curve $y = x^2 + 8x - 7$. **[4]**

(b) Write down the coordinates of the turning point of $y = x^2 + 8x - 7$. **[1]**

a) $x^2 + 8x - 7$

<u>completing the square</u>

$(x+4)^2 - (4)^2 - 7 \Rightarrow (x+4)^2 - 16 - 7$

$\Rightarrow (x+4)^2 - 23$

$\therefore$ the translation is $\begin{pmatrix} -4 \\ -23 \end{pmatrix}$

b) From the completed square

$(x+a)^2 + b$ the turning point $= (-a, b)$

$\therefore$ tp $= (-4, -23)$

3 **(a)** Express $\dfrac{1}{(x+2)(x+3)}$ in partial fractions. **[3]**

(b) Find $\displaystyle\int \frac{1}{(x+2)(x+3)}\,\mathrm{d}x$ in the form $\ln(\mathrm{f}(x))+c$, where c is the constant of integration and $\mathrm{f}(x)$ is a function to be determined. **[3]**

a) $\dfrac{1}{(x+2)(x+3)}$

$= \dfrac{A}{x+2} + \dfrac{B}{x+3} \equiv \dfrac{A(x+3)+B(x+2)}{(x+2)(x+3)}$

$\cdot \dfrac{A(x+3)+B(x+2)}{(x+2)(x+3)} = \dfrac{1}{(x+2)(x+3)}$

$\therefore A(x+3) + B(x+2) = 1$

let $x = -3$

$0 + B(-1) = 1$

$B = -1$

let $x = -2$

$A(1) = 1 \qquad A = 1$

$$\therefore \quad \frac{1}{x+2} - \frac{1}{x+3}$$

b) $$\int \frac{1}{x+2} - \frac{1}{x+3} \, dx$$

$$= \ln|x+2| - \ln|x+3| + C.$$

Using laws of logs

$$\ln A - \ln B = \ln\left(\frac{A}{B}\right)$$

$$\Rightarrow \ln\left|\frac{x+2}{x+3}\right| + C$$

$$\therefore \quad f(x) = \frac{x+2}{x+3}$$

4 In this question you must show detailed reasoning.

Show that $\frac{1}{\sqrt{10}+\sqrt{11}} + \frac{1}{\sqrt{11}+\sqrt{12}} + \frac{1}{\sqrt{12}+\sqrt{13}} = \frac{3}{\sqrt{10}+\sqrt{13}}$. **[3]**

1 Need to rationalise LHS

1 $\frac{1}{\sqrt{10}+\sqrt{11}} = \frac{1(\sqrt{10}-\sqrt{11})}{(\sqrt{10}+\sqrt{11})(\sqrt{10}-\sqrt{11})} = \frac{\sqrt{10}-\sqrt{11}}{\underbrace{10-11}_{-1}}$

$\Rightarrow -\sqrt{10}+\sqrt{11} \Rightarrow \sqrt{11}-\sqrt{10}$

2 $\frac{1}{\sqrt{11}+\sqrt{12}} = \frac{1(\sqrt{11}-\sqrt{12})}{(\sqrt{11}+\sqrt{12})(\sqrt{11}-\sqrt{12})} = \frac{\sqrt{11}-\sqrt{12}}{\underbrace{11-12}_{-1}}$

$\Rightarrow -\sqrt{11}+\sqrt{12} = \sqrt{12}-\sqrt{11}$

3 $\frac{1}{\sqrt{12}+\sqrt{13}} = \frac{1(\sqrt{12}-\sqrt{13})}{(\sqrt{12}+\sqrt{13})(\sqrt{12}-\sqrt{13})} = \frac{\sqrt{12}-\sqrt{13}}{\underbrace{12-13}_{-1}}$

$= -\sqrt{12}+\sqrt{13} = \sqrt{13}-\sqrt{12}$

1 + 2 + 3 $= \cancel{\sqrt{11}}-\sqrt{10}+(\cancel{\sqrt{12}}-\cancel{\sqrt{11}})+(\sqrt{13}-\cancel{\sqrt{12}})$

$= \sqrt{13}-\sqrt{10}$

② RHS

$$\frac{3}{\sqrt{10}+\sqrt{13}} = \frac{3(\sqrt{10}-\sqrt{13})}{(\sqrt{10}+\sqrt{13})(\sqrt{10}-\sqrt{13})} = \frac{3(\sqrt{10}-\sqrt{13})}{\underbrace{10-13}_{-3}}$$

$$\frac{\cancel{3}(\sqrt{10}-\sqrt{13})}{-\cancel{3}} = -\sqrt{10}+\sqrt{13}$$

$$= \sqrt{13}-\sqrt{10}$$

RHS = LHS : Both equations are equal.

5 A student's attempt to prove by contradiction that there is no largest prime number is shown below.

If there is a largest prime, list all the primes. Multiply all the primes and add 1. The new number is not divisible by any of the primes in the list and so it must be a new prime.

The proof is incorrect and incomplete.
Write a correct version of the proof. **[3]**

→ Suppose there is a large prime, p.

→ Multiply all the primes upto and including p, then add 1.

→ This number is not divisible by any of the primes $\therefore$ it is a prime.

→ This prime is larger than p

→ This is a contradiction, as the new prime is larger than p

→ $\therefore$ there is no largest prime

6 A circle has centre C (10, 4). The x-axis is a tangent to the circle, as shown in Fig. 6.

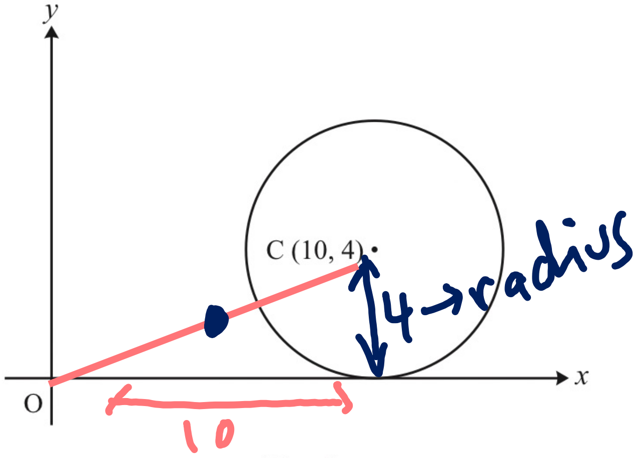


Fig. 6

(a) Find the equation of the circle. **[2]**

(b) Show that the line $y = x$ is not a tangent to the circle. **[4]**

(c) Write down the position vector of the midpoint of OC. **[1]**

a) $(x-a)^2 + (y-b)^2 = r^2$

$(a,b) \rightarrow$ centre.

By comparison, $(x-10)^2 + (y-4)^2 = 4^2$

$\Rightarrow (x-10)^2 + (y-4)^2 = 16$

b) $y = x$

Finding the point of intersection;

$(x-10)^2 + (x-4)^2 = 16$

$\Rightarrow x^2 - 20x + 100 + x^2 - 8x + \cancel{16} = \cancel{16}$

$\Rightarrow 2x^2 - 28x + 100 = 0$

Finding $b^2 - 4ac$

$(28)^2 - 4(2 \times 100)$

$= -16.$

$\therefore$ since $b^2 - 4ac < 0$

there are no point of intersection $\therefore$ $y = x$ can not be a tangent.

c) $10i + 4j \rightarrow OC$

$\therefore$ Midpoint $= \dfrac{10i + 4j}{2} = 5i + 2j$

7 In this question you must show detailed reasoning.

(a) Express $\ln 3 \times \ln 9 \times \ln 27$ in terms of $\ln 3$. **[2]**

(b) Hence show that $\ln 3 \times \ln 9 \times \ln 27 > 6$. **[2]**

a) $9 = 3^2$
$27 = 3^3$ } using this knowledge,

$\ln 3 \times \ln 3^2 \times \ln 3^3$

Using laws of logs:

$\log a^n = n\log a.$

$\therefore \Rightarrow \ln 3 \times 2\ln 3 \times 3\ln 3$

$\Rightarrow 6(\ln 3)^3$

b) we know that $e < 3$

$\therefore \ln 3 > 1$

$\therefore (\ln 3)^3 > 1$

$\therefore 6(\ln 3)^3 > 6 \times 1$

$\therefore 6(\ln 3)^3 > 6$ as required.

8 **In this question you must show detailed reasoning.**

A is the point (1, 0), B is the point (1, 1) and D is the point where the tangent to the curve $y = x^3$ at B crosses the x-axis, as shown in Fig. 8. The tangent meets the y-axis at E.

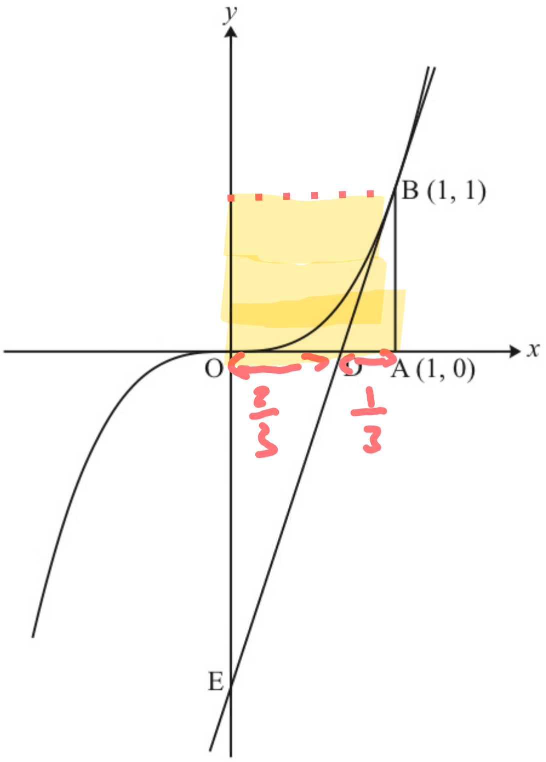


Fig. 8

(a) Find the area of triangle ODE. **[6]**

(b) Find the area of the region bounded by the curve $y = x^3$, the tangent at B and the y-axis. **[4]**

a) $y = x^3$

$\frac{dy}{dx} = 3x^2$

$\left.\frac{dy}{dx}\right|_{x=1} \Rightarrow 3(1)^2 = 3$

$\therefore$ Equation of the tangent;

$y - y_0 = m(x - x_0)$

$y - 1 = 3(x - 1)$

$y = 3x - 3 + 1$

$y = 3x - 2$

$\underline{X \text{ int}}$

$y = 0$

$\therefore \; 0 = 3x - 2$

$2 = 3x$

$x = \frac{2}{3} \quad \therefore \left(\frac{2}{3}, 0\right)$

$\underline{Y \text{ int}}$

$x = 0$

$\therefore \; y = -2$

$\therefore \; (0, -2)$

Area of a triangle

$= \frac{1}{2} \times \text{base} \times \text{height}$

$\therefore \; \frac{1}{2} \times \frac{2}{3} \times |-2| = \frac{2}{3} \text{ units}^2$

b) $\int_0^1 x^3 \, dx$

$= \left[\frac{x^4}{4}\right]_0^1 = \frac{1}{4}$

Area of rectangle highligted yellow.

$= 1 \times 1 = 1$

$\therefore$ Area bounded by y-axis;

$1 - \frac{1}{4} = \frac{3}{4}$ units^2

9 **In this question you must show detailed reasoning.**

The curve $xy + y^2 = 8$ is shown in Fig. 9.

$xy = 8 - y^2$

$x = \dfrac{8 - y^2}{y}$

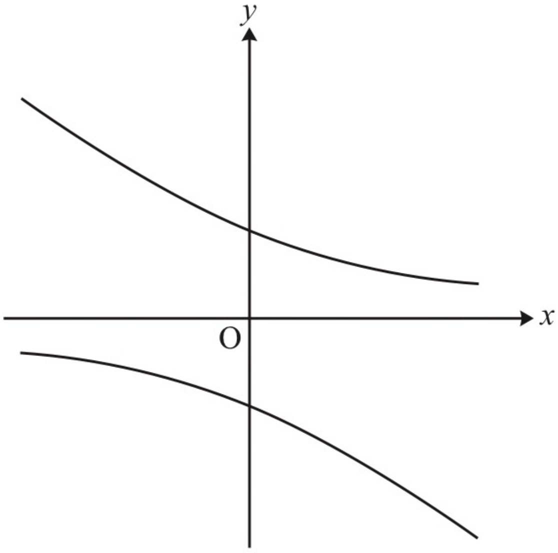


Fig. 9

$\therefore$ gradient of tangent $= -\frac{1}{2}$

Find the coordinates of the points on the curve at which the normal has gradient 2. **[6]**

Using implicit differentiation,

$x\dfrac{dy}{dx} + y + 2y \cdot \dfrac{dy}{dx} = 0$

As we stated above $\dfrac{dy}{dx}$ (ie gradient of tangent) $= -\frac{1}{2}$

$\therefore x(-\frac{1}{2}) + y + 2y(-\frac{1}{2}) = 0$

$-\dfrac{x}{2} + y - y = 0$ $\qquad \therefore x = 0$

when $x = 0$, from original eqn, $y^2 = 8$ $\therefore y = \pm\sqrt{8}$

· Co-ordinates $= (0, \sqrt{8})$ & $(0, -\sqrt{8})$

-

10 Show that $f(x) = \dfrac{e^x}{1+e^x}$ is an increasing function for all values of x. **[4]**

$$f(x) = \frac{e^x}{1+e^x}$$

$$\frac{dy}{dx} = ?$$

Using quotient rule;

$$\frac{vu' - uv'}{v^2}$$

$v = 1+e^x$ $\quad u = e^x$

$v' = e^x$ $\quad u' = e^x$

laws of indices

$$\frac{(1+e^x)e^x - [e^x(e^x)]}{(1+e^x)^2}$$

$$\Rightarrow \frac{e^x + e^{2x} - e^{2x}}{(1+e^x)^2} \qquad \Rightarrow \frac{e^x}{(1+e^x)^2}$$

- since $f'(x) > 0$

$f(x)$ is an increasing function for all values of x.

11 By using the substitution $u = 1 + \sqrt{x}$, find $\int \frac{x}{1+\sqrt{x}}\,dx$. **[6]**

$u = 1 + x^{1/2}$ — ①

$\frac{du}{dx} = \frac{1}{2}x^{-1/2}$ — ②

$\therefore \int \frac{x}{u}\,dx$

From ①

$(u-1)^2 = x$

From ②

$dx = \frac{2\,du}{x^{-1/2}} = 2\underbrace{x^{1/2}}\,du$

From ① $= u-1$.

$\therefore \int \frac{(u-1)^2}{u} \times 2(u-1)\,du$

$\Rightarrow 2\int \frac{(u-1)^3}{u}\,du \Rightarrow \frac{u^3 - 3u^2 + 3u - 1}{u}$

$\Rightarrow u^2 - 3u + 3 - \frac{1}{u}$

$\therefore 2\int u^2 - 3u + 3 - \frac{1}{u}\,du$

$$\Rightarrow 2\left[\frac{u^3}{3} - \frac{3u^2}{2} + 3u - \ln|u|\right]$$

$$\Rightarrow \frac{2u^3}{3} - 3u^2 + 6u - 2\ln|u| + C$$

But from ①

$$= \frac{2}{3}(1+\sqrt{x})^3 - 3(1+\sqrt{x})^2 + 6(1+\sqrt{x}) - 2\ln|1+\sqrt{x}| + c$$

Answer **all** the questions.

Section B (15 marks)

The questions in this section refer to the article on the Insert. You should read the article before attempting the questions.

12 Show that the equation of the line in Fig. C2 is $ry + hx = hr$, as given in line 24. **[2]**

$(r,0)$ & $(0,h)$ → co-ordinates . $m = \frac{h-0}{0-r} = -\frac{h}{r}$

$\therefore \; y = -\frac{h}{r}x + h$ $\times r$

$yr = -hx + hr$

$ry + hx = hr$ as required.

13 **(a)** **(i)** Show that the cross-sectional area in Fig. C3.2 is $\pi x(2r-x)$. **[2]**

(ii) Hence show that the cross-sectional area is $\frac{\pi r^2}{h^2}(h^2-y^2)$, as given in line 37. **[2]**

(b) Verify that the formula $\frac{\pi r^2}{h^2}(h^2-y^2)$ for the cross-sectional area is also valid for

(i) Fig. C3.1, **[1]**

(ii) Fig. C3.3. **[1]**

i) ① Area of $\square$

② $2 \times$ Area of the Δ

① $\underbrace{2x}_{\text{width}} \times \underbrace{L}_{\text{height}} = 2Lx$. But $L = \pi(r-x) = 2\pi x(r-x)$

② $2x$ = Diameter. radius $= \frac{2x}{2} = x$

∴ Area of semi circle $= \frac{\pi r^2}{2} = \frac{\pi}{2}(x)^2 = \frac{\pi x^2}{2}$

Since there are 2 semi-circles

$\Rightarrow 2 \times \frac{\pi x^2}{2} = \pi x^2$

$2\pi r x - 2\pi x^2 + \pi x^2$

$\Rightarrow 2\pi r x - \pi x^2 \quad \Rightarrow \pi x(2r-x)$ as required

ii) From $ry + hx = hr$

$$hx = hr - ry$$

$$x = \frac{hr - ry}{h}$$

· Replacing x into our statement gives:

$$\pi\left(\frac{hr-ry}{h}\right)\Big(\underbrace{2r - \left[\frac{hr-ry}{h}\right]}_{\text{simplifying this gives us;}}\Big)$$

↓

$$2r + \frac{-hr+ry}{h} \Rightarrow \frac{2hr - hr + ry}{h}$$

$$\Rightarrow \underbrace{\pi\left(\frac{hr-ry}{h}\right)\left(\frac{hr+ry}{h}\right)}_{\text{difference between 2 squares}} \Rightarrow \frac{\pi}{h^2} \cdot r(h-y) \cdot r(h+y)$$

$$\Rightarrow \frac{\pi r^2}{h^2}(h^2 - y^2)$$ as required.

bi) $y=0$ $\therefore$ $\frac{\pi r^2}{\cancel{h^2}}(\cancel{h^2}-0) = \pi r^2$ which is true.

ii) $y=h$ $\therefore$ $\frac{\pi r^2}{h^2}(h^2-(h)^2) = 0$ which is also true.

14 **(a)** Express $\lim_{\delta y \to 0} \sum_{0}^{h} (h^2 - y^2)\delta y$ as an integral. **[1]**

(b) Hence show that $V = \frac{2}{3}\pi r^2 h$, as given in line 41. **[3]**

a) $\int_0^h (h^2 - y^2)\, dy$

b) $V = \frac{\pi r^2}{h^2} \int_0^h (h^2 - y^2)\, dy$

$\frac{\pi r^2}{h^2}\left[h^2 y - \frac{y^3}{3}\right]_0^h \Rightarrow \frac{\pi r^2}{h^2}\left[\left[h^3 - \frac{h^3}{3}\right] - [0 - 0]\right]$

$= \frac{\pi r^2}{h^2}\left[\frac{2h^3}{3}\right] = \frac{2\pi r^2 h}{3} = \frac{2}{3}\pi r^2 h$ as required.

15 A typical tube of toothpaste measures 5.4 cm across the straight edge at the top and is 12 cm high. It contains 75 ml of toothpaste so it needs to have an internal volume of $75\,\text{cm}^3$.

Comment on the accuracy of the formula $V = \frac{2}{3}\pi r^2 h$, as given in line 41, for the volume in this case. **[3]**

$\pi r = 5.4$

$\therefore\ r = \frac{5.4}{\pi} = 1.718... \approx 1.72$

$\therefore\ V = \frac{2\pi r^2 h}{3} = \frac{2}{3}\pi\left[\frac{5.4}{\pi}\right]^2 \times 12$

$= 74.2$

→ The volume / radius is too small but it is close enough.

→ Allowing for approximation, the formula seems correct

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Friday 14 June 2019 – Afternoon

A Level Mathematics B (MEI)

H640/03 Pure Mathematics and Comprehension

Insert

Time allowed: 2 hours

INFORMATION
- This Insert contains the article for Section B.
- This document consists of **4** pages.

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Modelling a tube

Products such as toothpaste and hand cream are often sold in tubes which have a circular cross-section at the end which has the opening for the product to be dispensed. The other end of the tube is closed and is a straight line. The front view and side view of such a tube are shown in Fig. C1. The circular end will be defined to be the bottom end of the tube and the straight line end will be defined to be the top end.

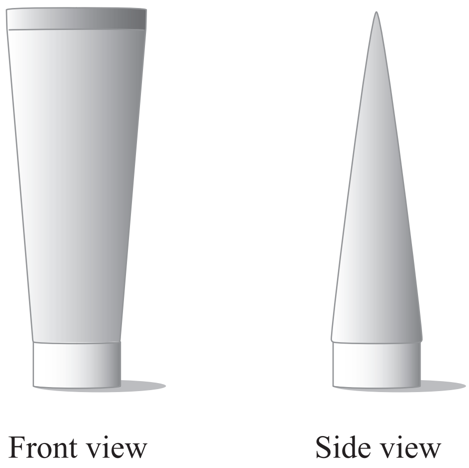

Fig. C1

There is no simple formula for the volume of a tube of this shape, but a good approximation can be derived using mathematical modelling.

The cross-section at the bottom of the tube is a circle; the cross-section at the top is a straight line. Observation of tubes suggests that they are made by starting with a cylinder and closing one end by bringing the sides together in a straight line. This means that the tube will have a volume smaller than the cylinder that was used when making it. If the base radius of the tube is r, the height is h and the volume is V then

$$V < \pi r^2 h.$$

Modelling assumptions

The following table lists the modelling assumptions which will be made, together with some comments justifying each of them.

Modelling assumption	Comments
The perimeter of the cross-section of the tube is constant all the way up.	This follows from starting with a cylinder to make the tube.
The nozzle at the bottom of the tube and the cap will be ignored.	Experience suggests that the nozzle and cap are not filled with the product when the tube is first opened so their volumes are not relevant.
The front width of the tube increases at a constant rate from the bottom end to the top end.	Observation suggests that this is a good approximation.
The side width of the tube decreases at a constant rate from the bottom end to the top end.	This situation is shown in Fig. C2; observation suggests that this is a close approximation for tubes of typical sizes.

Modelling the cross-section

Taking the y-axis as the axis of symmetry of the tube and looking at the tube from the side, as shown in Fig. C2, means that the side width of the tube is $2x$ at height y.

When $y=0, x=r$ and when $y=h, x=0$.

Assuming that the relationship between x and y is linear means that the side width decreases at a constant rate as y increases; this leads to $ry+hx=hr$.

The cross-section at the bottom of the tube is a circle, as shown in Fig. C3.1; at the top of the tube, the cross-section is a line, as shown in Fig. C3.3.

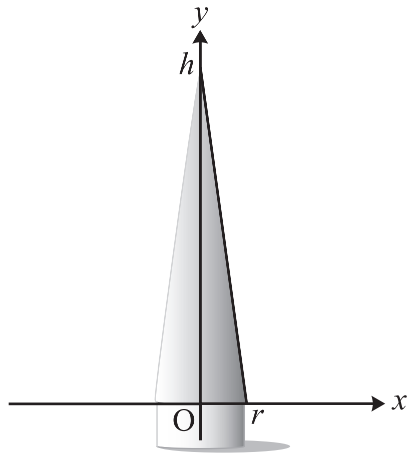

Fig. C2

Area of a circle $= \pi r^2$

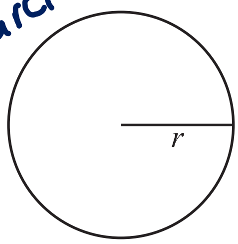

Fig. C3.1

$l = \pi(r-x)$

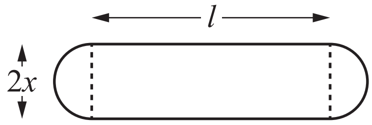

Fig. C3.2

Area of a line = 0

Fig. C3.3

The exact ‘oval’ shape of the cross-section at intermediate points is not easy to determine, so a simple approximation for the shape is used.

When the width of the tube is $2x$, the cross-section will be modelled as a rectangle with semicircular ends, as shown in Fig. C3.2. The radius of the semicircular ends is x. To ensure that the total perimeter of the cross-section is a constant, the length, l, of the rectangular part of the cross-section is given by $l=\pi(r-x)$. It can be shown that this ensures that the front width of the tube increases at a constant rate as y increases, as required by the modelling assumptions.

Calculating the volume

Finding the area of the cross-section shown in Fig. C3.2 and using $ry+hx=hr$ gives the cross-sectional area in terms of y as $\frac{\pi r^2}{h^2}(h^2-y^2)$.

Imagine slicing the tube into thin horizontal slices, with cross-section as shown in Fig. C3.2 and thickness δy. The volume of the tube is given by $\sum_0^h \frac{\pi r^2}{h^2}(h^2-y^2)\delta y$; since r and h are constants for the tube, this can be written as $\frac{\pi r^2}{h^2}\sum_0^h (h^2-y^2)\delta y$.

Taking the limit as $\delta y \to 0$ and evaluating the resulting integral gives $V=\frac{2}{3}\pi r^2 h$. This is less than the volume of the cylinder, $\pi r^2 h$, as expected.