Oxford Cambridge and RSA

# A Level Mathematics B (MEI)

## H640/03 Pure Mathematics and Comprehension
Question Paper

## Friday 15 June 2018 – Afternoon
**Time allowed: 2 hours**

**You must have:**
- Printed Answer Booklet
- Insert

**You may use:**
- a scientific or graphical calculator

Model Answers

**INSTRUCTIONS**
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Complete the boxes provided on the Printed Answer Booklet with your name, centre number and candidate number.
- Answer **all** the questions.
- **Write your answer to each question in the space provided in the Printed Answer Booklet.** If additional space is required, you should use the lined page(s) at the end of the Printed Answer Booklet. The question number(s) must be clearly shown.
- Do **not** write in the barcodes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

**INFORMATION**
- The total mark for this paper is **75**.
- The marks for each question are shown in brackets **[ ]**.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is used. You should communicate your method with correct reasoning.
- The Printed Answer Booklet consists of **20** pages. The Question Paper consists of **8** pages.

**Section A** (60 marks)

**1** Triangle ABC is shown in Fig. 1.

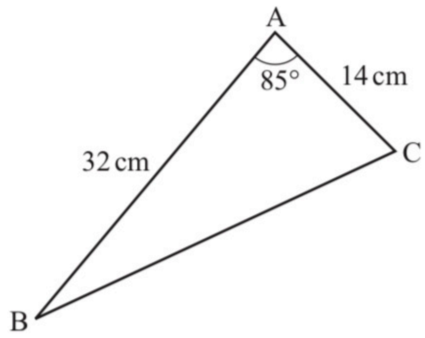
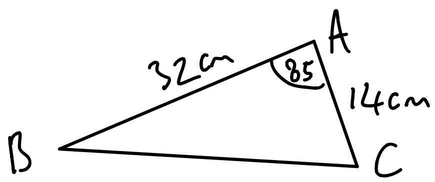



**Fig. 1**

Find the perimeter of triangle ABC. **[3]**




Cosine rule: $a^2 = b^2 + c^2 - 2bc \cdot \cos(A)$

We have $b=32$, $c=14$, $A=85$

Hence,

$BC^2 = 32^2 + 14^2 - 2(32)(14)\cos(85)$

$BC^2 = 1141.9$

$BC = 33.79$

So perimeter $= 32\text{cm} + 14\text{cm} + 33.79\text{cm}$

$= 79.79\text{cm}$

**2** The curve $y = x^3 - 2x$ is translated by the vector $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$. Write down the equation of the translated curve. **[2]**

If $y = f(x)$ the translation is $f(x-1) - 4$

$f(x-1) - 4 = (x-1)^3 - 2(x-1) - 4$

**3** Fig. 3 shows a circle with centre O and radius 1 unit. Points A and B lie on the circle with angle AOB $= \theta$ radians. C lies on AO, and BC is perpendicular to AO.

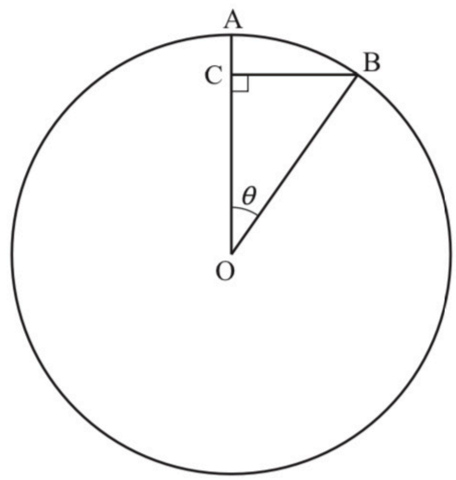



**Fig. 3**

Show that, when $\theta$ is small, AC $\approx \frac{1}{2}\theta^2$. **[2]**

$AC = AO - CO$

$= 1 - \cos\theta$ , when $\theta$ is small $\cos\theta \approx 1 - \frac{\theta^2}{2}$

$= 1 - (1 - \frac{\theta^2}{2})$

$= \frac{\theta^2}{2}$

**4 In this question you must show detailed reasoning.**

A curve has equation $y = x - 5 + \dfrac{1}{x-2}$. The curve is shown in Fig. 4.

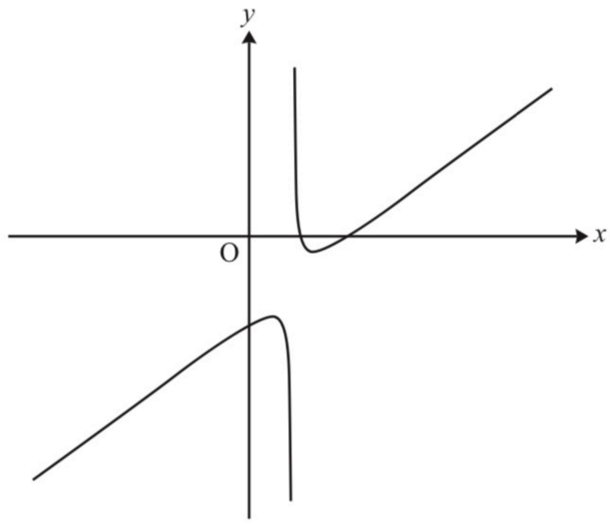



**Fig. 4**

**(i)** Determine the coordinates of the stationary points on the curve. **[5]**

$y = x - 5 + \dfrac{1}{x-2}$ , $\dfrac{dy}{dx} = 1 - \dfrac{1}{(x-2)^2}$

At stationary points $\dfrac{dy}{dx} = 0$

$0 = 1 - \dfrac{1}{(x-2)^2}$

$1 = \dfrac{1}{(x-2)^2}$

$(x-2)^2 = 1$

$x - 2 = \pm 1$

$x = 2 \pm 1$

$x = 1$ or $3$

When $x=1$ , $y = 1 - 5 + \dfrac{1}{-1} = -5$

When $x=3$ , $y = 3 - 5 + \dfrac{1}{1} = -1$

The coordinates are $(1, -5)$ and $(3, -1)$

**(ii)** Determine the nature of each stationary point. **[3]**

$$\frac{d^2y}{dx^2} = \frac{2}{(x-2)^3}$$

When $x=1$, $\frac{d^2y}{dx^2} = \frac{2}{-1} = -2 < 0$

When $x=3$, $\frac{d^2y}{dx^2} = 2 > 0$

So $x=1$ is a maximum and $x=3$ is a minimum.

**(iii)** Write down the equation of the vertical asymptote. **[1]**

$x=2$

**(iv)** Deduce the set of values of $x$ for which the curve is concave upwards. **[1]**

$x>2$

**5** A social media website launched on 1 January 2017. The owners of the website report the number of users the site has at the start of each month. They believe that the relationship between the number of users, $n$, and the number of months after launch, $t$, can be modelled by $n = a \times 2^{kt}$ where $a$ and $k$ are constants.

**(i)** Show that, according to the model, the graph of $\log_{10} n$ against $t$ is a straight line. **[2]**

$n = a \times 2^{kt}$

$\log_{10}(n) = \log_{10}(a \times 2^{kt})$

$\log_{10}(n) = \log_{10}(a) + kt\log_{10}(2)$.

This is in the form of a straight line $y = mx + c$ with $y = \log_{10}(n)$ and $x = t$

**(ii)** Fig. 5 shows a plot of the values of $t$ and $\log_{10}n$ for the first seven months. The point at $t = 1$ is for 1 February 2017, and so on.

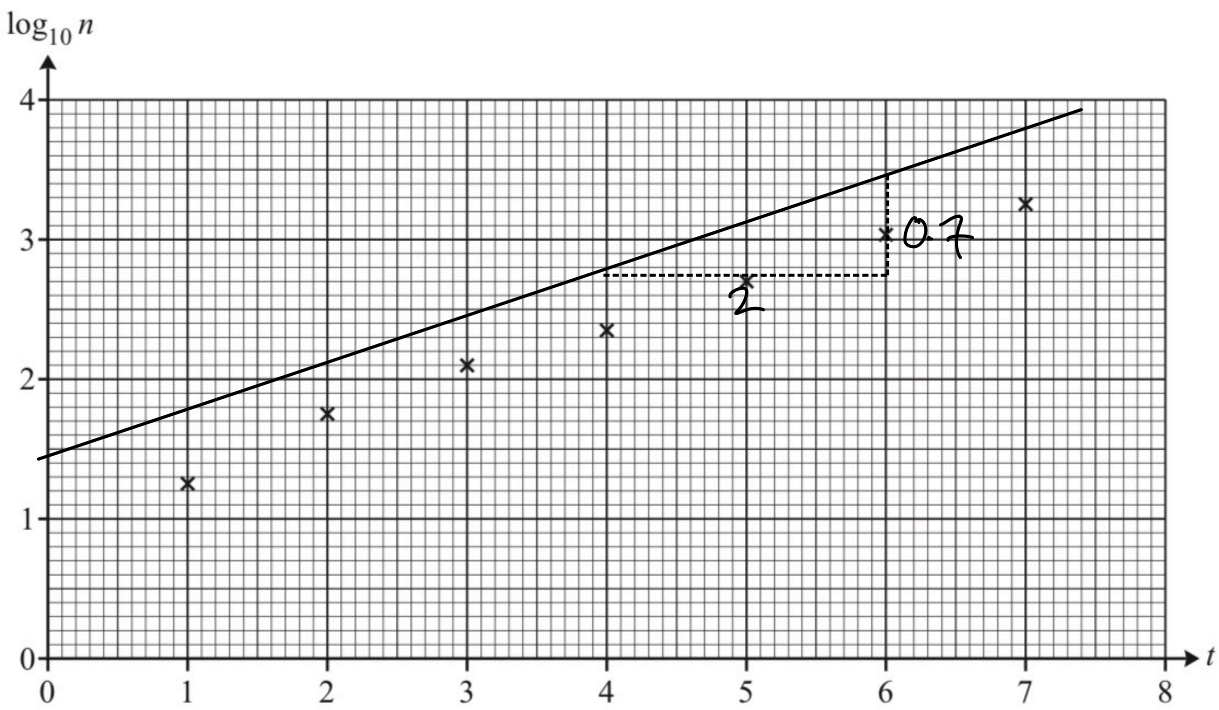



**Fig. 5**

Find estimates of the values of $a$ and $k$. **[4]**

$C = \log_{10}(a) \approx 1$

$\Rightarrow a \approx 10$

$m = k\log_{10}(2) \approx \frac{0.7}{2}$

$k = \frac{0.7}{2\log_{10}(2)}$

$k = 1.16$

So $a \approx 10$, $k \approx 1.16$

**(iii)** The owners of the website wanted to know the date on which they would report that the website had half a million users. Use the model to estimate this date. **[4]**

When $n = 500{,}000$

$500{,}000 = 10 \times 2^{1.16 \times t}$

$50{,}000 = 2^{1.16 \times t}$

$1.16t = \log_2(50{,}000)$

$t = \frac{15.6096}{1.16}$

$t = 13.46$

$t = 13$

The date when $t=13$ is 01/02/18. But they will first report it on 01/03/18 as this will be the next time they'll check the number.

**(iv)** Give a reason why the model may not be appropriate for large values of $t$. **[1]**

You could be extrapolating the data as it is only valid for up to $t=7$.

**6** Find the constant term in the expansion of $\left(x^2 + \frac{1}{x}\right)^{15}$. **[2]**

The constant term is when you have $(x^2)^n \times \left(\frac{1}{x}\right)^{2n}$ so they cancel. $n=5$ is the only way to get this,

$$\binom{15}{5}(x^2)^5\left(\frac{1}{x}\right)^{10} = 3003$$

**7 In this question you must show detailed reasoning.**

Fig. 7 shows the curve $y = 5x - x^2$.

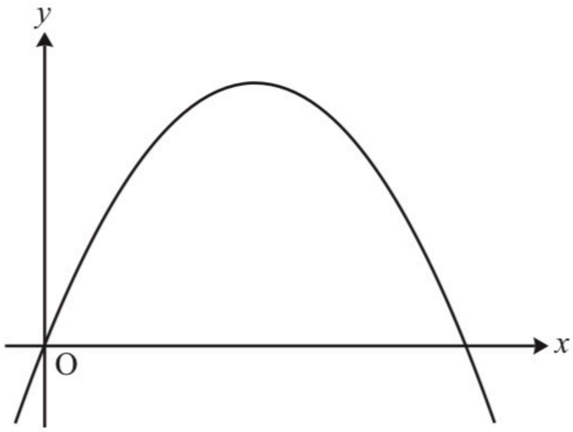



**Fig. 7**

The line $y = 4 - kx$ crosses the curve $y = 5x - x^2$ on the $x$-axis and at one other point.

Determine the coordinates of this other point. **[8]**

$y = 5x - x^2 \quad y = 4 - kx$

They cross on the $x$-axis which is where $y=0$

$0 = 5x - x^2$

$0 = x(5-x) \quad$ so $x = 0$ or $5$

The coordinates at which they cross cannot be $(0,0)$ since $y = 4 - kx$ does not pass through the origin. Hence they cross at $(5,0)$, we use this to find $k$.

$0 = 4 - 5k$

$k = \frac{4}{5}$

Now set them equal to find the other point where they cross:

$5x - x^2 = 4 - \frac{4}{5}x$

$0 = 5x^2 - 24x + 20$

$0 = (5x-4)(x-5)$

$x = \frac{4}{5}$

$y = 4 - \frac{4}{5}\left(\frac{4}{5}\right) = 4 - \frac{16}{25} = \frac{84}{25}$

$\therefore$ They meet at $\left(\frac{4}{5}, \frac{84}{25}\right)$

**8** A curve has parametric equations $x = \dfrac{t}{1+t^3}$, $y = \dfrac{t^2}{1+t^3}$, where $t \neq -1$.

**(i)** **In this question you must show detailed reasoning.**

Determine the gradient of the curve at the point where $t = 1$. **[5]**

$x = \dfrac{t}{1+t^3} \qquad y = \dfrac{t^2}{1+t^3}$

$\dfrac{dx}{dt} = \dfrac{(1+t^3) - t(3t^2)}{(1+t^3)^2} = \dfrac{1-2t^3}{(1+t^3)^2}$

$\dfrac{dy}{dt} = \dfrac{2t(1+t^3) - t^2(3t^2)}{(1+t^3)^2} = \dfrac{2t - t^4}{(1+t^3)^2}$

$\dfrac{dy}{dx} = \dfrac{dy}{dt} \div \dfrac{dx}{dt} = \dfrac{2t - t^4}{1-2t^3}$

When $t=1$, $\dfrac{dy}{dx} = \dfrac{2-1}{1-2} = \dfrac{1}{-1} = -1$

**(ii)** Verify that the cartesian equation of the curve is $x^3 + y^3 = xy$. **[3]**

$x^3 + y^3 = xy$

$\dfrac{t^3}{(1+t^3)^3} + \dfrac{t^6}{(1+t^3)^3} = \left(\dfrac{t}{1+t^3}\right)\left(\dfrac{t^2}{1+t^3}\right)$

$\dfrac{t^3(1+t^3)}{(1+t^3)^3} = \dfrac{t^3}{(1+t^3)^2}$

$\dfrac{t^3}{(1+t^3)^2} = \dfrac{t^3}{(1+t^3)^2}$

**9** The function $f(x) = \dfrac{e^x}{1-e^x}$ is defined on the domain $x \in \mathbb{R}$, $x \neq 0$.

**(i)** Find $f^{-1}(x)$. **[3]**

$$f(x) = \frac{e^x}{1-e^x}$$

Let $x = \dfrac{e^y}{1-e^y}$

$$x - xe^y = e^y$$
$$x = e^y(1+x)$$
$$e^y = \frac{x}{1+x}$$
$$y = \ln\left(\frac{x}{1+x}\right)$$

$$\therefore f^{-1}(x) = \ln\left(\frac{x}{1+x}\right)$$

**(ii)** Write down the range of $f^{-1}(x)$. **[1]**

$f^{-1}(x) \in \mathbb{R}$ , $f^{-1}(x) \neq 0$

**10** Point A has position vector $\begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$ where $a$ and $b$ can vary, point B has position vector $\begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$ and point C has position vector $\begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$. ABC is an isosceles triangle with AC = AB.

**(i)** Show that $a - b + 1 = 0$. **[4]**

$$\overrightarrow{AC} = C - A = \begin{pmatrix} 2-a \\ 4-b \\ 2 \end{pmatrix}$$

$$\overrightarrow{AB} = B - A = \begin{pmatrix} 4-a \\ 2-b \\ 0 \end{pmatrix}$$

$$AC = AB$$

$$(2-a)^2 + (4-b)^2 + 4 = (4-a)^2 + (2-b)^2$$

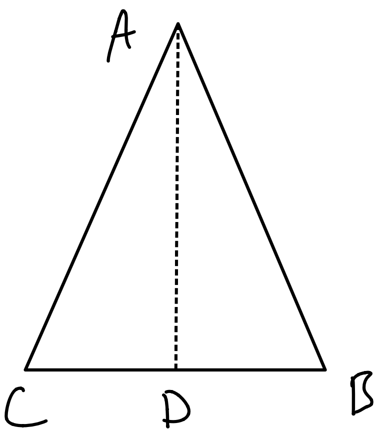
$$\cancel{4} - 4a + \cancel{a^2} + \cancel{16} - 8b + \cancel{b^2} + 4 = \cancel{16} - 8a + \cancel{a^2} + \cancel{4} - 4b + \cancel{b^2}$$

$$-4a + 8a - 8b + 4b + 4 = 0$$

$$4a - 4b + 4 = 0$$

$$a - b + 1 = 0$$

**(ii)** Determine the position vector of A such that triangle ABC has minimum area. **[6]**




$$D = \text{midpoint of } \overrightarrow{BC} = \frac{1}{2}\begin{pmatrix} 4+2 \\ 2+4 \\ 0+2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$$

$$\text{Area} = \tfrac{1}{2} \times AD \times CB$$

$$\overrightarrow{AD} = D - A = \begin{pmatrix} 3-a \\ 3-b \\ 1-0 \end{pmatrix}$$

$$= \begin{pmatrix} 3-a \\ 2-a \\ 1 \end{pmatrix} \quad \text{using } b = a+1 \text{ from part (i) above}$$

$$AD^2 = (3-a)^2 + (2-a)^2 + 1 = 9 - 6a + a^2 + 4 - 4a + a^2 + 1 = 2a^2 - 10a + 14$$

$$\overrightarrow{CB} = B - C = \begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix}, \quad CB^2 = 4 + 4 + 4 = 12$$

$$CB = \sqrt{12} = 2\sqrt{3}$$

Hence, area $= \frac{1}{2} \times 2\sqrt{3} \times \sqrt{2a^2-10a+14} = \sqrt{3} \times \sqrt{2} \times \sqrt{a^2-5a+7}$
$= \sqrt{6}\sqrt{a^2-5a+7}$

we want to minimise by completing the square,
$= \sqrt{6}\sqrt{(a-2.5)^2-6.25+7}$
$= \sqrt{6}\sqrt{(a-2.5)^2+0.75}$

So this is minimised when $a=2.5$.
$\therefore$ A has position vector $\begin{pmatrix} 2.5 \\ 2.5+1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2.5 \\ 3.5 \\ 0 \end{pmatrix}$

**The questions in this section refer to the article on the Insert. You should read the article before attempting the questions.**

**11** Line 8 states that $\frac{a+b}{2} \geqslant \sqrt{ab}$ for $a$, $b \geqslant 0$. Explain why the result cannot be extended to apply in each of the following cases.

**(i)** One of the numbers $a$ and $b$ is positive and the other is negative. **[1]**

ab is negative so you cannot find the geometric mean.

**(ii)** Both numbers $a$ and $b$ are negative. **[1]**

The arithmetic mean will be less than the geometric mean.

**12** Lines 5 and 6 outline the stages in a proof that $\frac{a+b}{2} \geqslant \sqrt{ab}$. Starting from $(a-b)^2 \geqslant 0$, give a detailed proof of the inequality of arithmetic and geometric means. **[3]**

$(a-b)^2 \geq 0$
$a^2-2ab+b^2 \geq 0$
$a^2+b^2 \geq 2ab$
$a^2+2ab+b^2 \geq 4ab$
$(a+b)^2 \geq 4ab$
$(a+b) \geq \sqrt{4ab} = 2\sqrt{ab}$
$\frac{a+b}{2} \geq \sqrt{ab}$

**13** Consider a geometric sequence in which all the terms are positive real numbers. Show that, for any three consecutive terms of this sequence, the middle one is the geometric mean of the other two. **[3]**

Let the terms be $a, ar, ar^2$

Then the geometric mean of $a$ and $ar^2$ is $\sqrt{a^2r^2} = ar$, which is the middle term.

**14** **(i)** In Fig. C1.3, angle CBD = $\theta$. Show that angle CDA is also $\theta$, as given in line 23. **[2]**

Angles in a triangle add up to $180°$, so

$\angle CDB = 180 - 90 - \theta$

$= 90 - \theta$

$\angle ADB$ is $90°$ so $\angle CDA = \theta$

**(ii)** Prove that $h = \sqrt{ab}$, as given in line 24. **[2]**

From triangle CBD, $\tan\theta = \frac{h}{b}$

From triangle ACD, $\tan\theta = \frac{a}{h}$

$\Rightarrow \frac{h}{b} = \frac{a}{h}$

$\Rightarrow h^2 = ab$

$h = \sqrt{ab}$

**15** It is given in lines 31–32 that the square has the smallest perimeter of all rectangles with the same area. Using this fact, prove by contradiction that among rectangles of a given perimeter, $4L$, the square with side $L$ has the largest area. **[3]**

Suppose that there is a rectangle of perimeter $4L$ that has an area larger than the square.

Using lines 31–32 there is a square with the same area as this rectangle but with a smaller perimeter, meaning its sides are less than $L$.

Therefore a square that has side length $L$ has perimeter $4L$ and an area larger than the rectangle. This is a contradiction which means the square must have the largest area of all rectangles with the same perimeter.