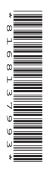


Monday 19 October 2020 – Afternoon

A Level Mathematics B (MEI)

H640/03 Pure Mathematics and Comprehension Insert

Time allowed: 2 hours



INSTRUCTIONS

• Do not send this Insert for marking. Keep it in the centre or recycle it.

INFORMATION

- This Insert contains the article for Section B.
- This document has 4 pages.

PMT

Which is bigger?

Which is bigger: π^{e} or e^{π} ? Using a calculator confirms that e^{π} is the larger, but how can this be proved without the use of a calculator?

Simpler examples

It is often helpful in mathematics to consider simpler examples. It is easy to work out that $3^4 > 4^3$. In the expression 3^4 , 3 is the base and 4 is the exponent. Working with integers greater than 1, it is easy to find many examples where $a^b > b^a$ if a < b. That is, using the smaller base and the larger exponent gives the larger result. This might lead us to conjecture that $a^b > b^a$ if a < b and both a and b are integers greater than 1. However, it is also possible to find counter examples to this conjecture.

Exponents can also be rational numbers, and in general $x^{\frac{p}{q}}$ denotes $(\sqrt[q]{x})^p$ where *p* and *q* are integers 10 and *q* is positive. So, any rational power of a positive number, *x*, can be defined. However, both e and π are irrational numbers. Considering the original question about π^e and e^{π} raises the issue of what is meant by an irrational power of a number.

Extending the definition of power to irrational numbers

What, for example, is meant by 2^{π} ?

An irrational number corresponds to a non-recurring infinite decimal. Rounding the decimal gives a rational approximation to the irrational number. For example, the following sequence gives increasingly accurate approximations to π .

3, 3.1, 3.14, 3.142, 3.1416, 3.14159, ...

Using a spreadsheet gives a sequence of approximations to 2^{π} , as shown in Fig. C1. The limit of this 20 sequence of approximations is the value of 2^{π} . This limit cannot be evaluated with a spreadsheet but it is, in principle, possible to find the value to any required degree of accuracy.

	Α	В	
1	k	2 ^{<i>k</i>}	
2	3	8	
3	3.1	8.574188	
4	3.14	8.815241	
5	3.142	8.82747	
6	3.1416	8.825023	
7	3.14159	8.824962	

Fig. C1

 2^x and x^2 are increasing functions of x for x > 0 and this allows us to deduce that $\pi^2 > 2^{\pi}$, as follows.

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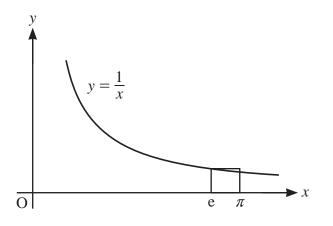
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We know that π is between 3 and 3.142

 $\pi < 3.142 \Rightarrow 2^{\pi} < 2^{3.142} = 8.82747$ $\pi > 3 \Rightarrow \pi^2 > 3^2 = 9$ So $\pi^2 > 9 > 8.82747 > 2^{\pi}$ Hence $\pi^2 > 2^{\pi}$

Which is bigger: π^{e} or e^{π} ?

An indirect method, using calculus, enables us to prove that e^{π} is larger than π^{e} . Fig. C2 shows the curve $y = \frac{1}{r}$ in the first quadrant together with the rectangle with vertices at the points (e, 0), $\left(e,\frac{1}{e}\right),\left(\pi,\frac{1}{e}\right)$ and $(\pi, 0)$. We use the fact that the area under the curve between e and π is less than the area of this rectangle.





The area of the rectangle is $\frac{1}{e}(\pi - e)$

$$\int_{e}^{\pi} \frac{1}{x} dx < \frac{1}{e}(\pi - e)$$
$$\ln \pi - 1 < \frac{\pi}{e} - 1$$
$$\ln \pi < \frac{\pi}{e}$$

 e^x is an increasing function for all values of x

hence
$$\pi < e^{\frac{\pi}{e}}$$
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Assuming that the usual rules of indices apply to irrational powers of irrational numbers, raising both sides of the inequality to the power e gives the desired result.

Using a similar method, it can be shown that $e^a > a^e$ for any positive number $a \neq e$.

An alternative method for showing that $e^a > a^e$ for any positive number a is to show that the only stationary point on the curve $y = \frac{\ln x}{x}$ (a maximum) occurs where x = e. 45

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