Oxford Cambridge and RSA

Monday 13 May 2019 – Afternoon

AS Level Further Mathematics B (MEI)

Y410/01 Core Pure

Time allowed: 1 hour 15 minutes

Model Answers

You must have:
- Printed Answer Booklet
- Formulae Further Mathematics B (MEI)

You may use:
- a scientific or graphical calculator

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INSTRUCTIONS
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Answer **all** the questions.
- **Write your answer to each question in the space provided in the Printed Answer Booklet.** If additional space is required, you should use the lined page(s) at the end of the Printed Answer Booklet. The question number(s) must be clearly shown.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION
- The total number of marks for this paper is **60**.
- The marks for each question are shown in brackets **[ ]**.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is used. You should communicate your method with correct reasoning.
- The Printed Answer Booklet consists of **12** pages. The Question Paper consists of **4** pages.

Answer **all** the questions

1 In this question you must show detailed reasoning.

Find $\sum_{r=1}^{100}\left(\frac{1}{r}-\frac{1}{r+2}\right)$, giving your answer correct to 4 decimal places. **[3]**

$$\sum_{r=1}^{100}\left(\frac{1}{r}-\frac{1}{r+2}\right)=\frac{1}{1}-\frac{1}{3}+\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{5}+\dots+\frac{1}{98}-\frac{1}{100}+\frac{1}{99}-\frac{1}{101}+\frac{1}{100}-\frac{1}{102}$$

Substitute $r=1$ to 100 and see what cancels out.

cancels out

$$=1+\frac{1}{2}-\frac{1}{101}-\frac{1}{102}$$

$$=1.4803 \text{ (4dp)}$$

2 The roots of the equation $3x^2-x+2=0$ are α and β.
Find a quadratic equation with integer coefficients whose roots are $2\alpha-3$ and $2\beta-3$. **[3]**

$w=2x-3 \Rightarrow x=\frac{w+3}{2}$. Substitute this into $3x^2-x+2=0$

$$\Rightarrow 3\left(\frac{w+3}{2}\right)^2-\frac{w+3}{2}+2=3\left(\frac{w^2+6w+9}{4}\right)-\frac{w+3}{2}+2$$

$$=\frac{3w^2+18w+27}{4}-\frac{2w+6}{4}+\frac{8}{4}=\frac{3w^2+16w+29}{4}.$$

$$\therefore \frac{3w^2+16w+29}{4}=0 \Rightarrow 3w^2+16w+29=0$$

3 In this question you must show detailed reasoning.

A and **B** are matrices such that $\mathbf{B}^{-1}\mathbf{A}^{-1}=\begin{pmatrix}2 & 1\\ -1 & 1\end{pmatrix}$.

(a) Find **AB**. **[3]**

(b) Given that $\mathbf{A}=\begin{pmatrix}\frac{1}{3} & 1\\ 0 & 1\end{pmatrix}$, find **B**. **[3]**

a) $(AB)^{-1} = B^{-1}A^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$ so $AB = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}^{-1}$.

Determinant of $\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} = (2 \times 1) - (-1 \times 1) = 3$.

$\Rightarrow AB = \frac{1}{3}\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$ OR $\begin{pmatrix} 1/3 & -1/3 \\ 1/3 & 2/3 \end{pmatrix}$

b) $A = \begin{pmatrix} 1/3 & 1 \\ 0 & 1 \end{pmatrix} \Rightarrow \det A = \frac{1}{3}$ so $A^{-1} = 3\begin{pmatrix} 1 & -1 \\ 0 & 1/3 \end{pmatrix}$

$= \begin{pmatrix} 3 & -3 \\ 0 & 1 \end{pmatrix}$

$B = A^{-1} \times AB = 3\begin{pmatrix} 1 & -1 \\ 0 & 1/3 \end{pmatrix} \times \frac{1}{3}\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$

$= \begin{pmatrix} 1 & -1 \\ 0 & 1/3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & -3 \\ 1/3 & 2/3 \end{pmatrix}$

4 **(a)** Find $\mathbf{M}^{-1}$, where $\mathbf{M} = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ -2 & 1 & 2 \end{pmatrix}$. **[1]**

(b) Hence find, in terms of the constant k, the point of intersection of the planes

$$x + 2y + 3z = 19,$$
$$-x + y + 2z = 4,$$
$$-2x + y + 2z = k.$$ **[3]**

(c) **In this question you must show detailed reasoning.**

Find the acute angle between the planes $x + 2y + 3z = 19$ and $-x + y + 2z = 4$. **[4]**

a) $M^{-1} = \begin{pmatrix} 0 & 1 & -1 \\ 2 & -8 & 5 \\ -1 & 5 & -3 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ -2 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 19 \\ 4 \\ k \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ 2 & -8 & 5 \\ -1 & 5 & -3 \end{pmatrix}\begin{pmatrix} 19 \\ 4 \\ k \end{pmatrix}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4-k \\ 6+5k \\ 1-3k \end{pmatrix}$ $\therefore$ Point of intersection is $x = 4-k,\ y = 6+5k,\ z = 1-3k$

c) Normals are $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\begin{pmatrix}-1\\1\\2\end{pmatrix}$

$\left|\begin{matrix}1\\2\\3\end{matrix}\right| = \sqrt{14}$ $\left|\begin{matrix}-1\\1\\2\end{matrix}\right| = \sqrt{6}$ $\begin{pmatrix}1\\2\\3\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\2\end{pmatrix} = 7$

$\cos\theta = \dfrac{7}{\sqrt{6}\sqrt{14}} = 0.764 \Rightarrow \theta = 40.2^\circ$

5 Prove by induction that, for all positive integers n, $\displaystyle\sum_{r=1}^{n}\frac{1}{3^r} = \frac{1}{2}\left(1-\frac{1}{3^n}\right)$. **[6]**

When $n=1$, LHS $= \frac{1}{3}$ and RHS $= \frac{1}{2}\left(1-\frac{1}{3}\right)$
$= \frac{1}{2}\times\frac{2}{3} = \frac{1}{3}$. As LHS = RHS, this is true for $n=\underline{1}$.

Assume this is true for $n=k$ so
$\displaystyle\sum_{r=1}^{k}\frac{1}{3^r} = \frac{1}{2}\left(1-\frac{1}{3^k}\right)$.

For $n=k+1$: $\displaystyle\sum_{r=1}^{k+1}\frac{1}{3^r} = \sum_{r=1}^{k}\frac{1}{3^r} + \frac{1}{3^{k+1}}$

$= \frac{1}{2}\left(1-\frac{1}{3^k}\right) + \frac{1}{3^{k+1}} = \frac{1}{2}\left(1-\frac{1}{3^k}+\frac{2}{3^{k+1}}\right)$

$= \frac{1}{2}\left(1-\frac{3}{3^{k+1}}+\frac{2}{3^{k+1}}\right) = \frac{1}{2}\left(1-\frac{1}{3^{k+1}}\right)$

Therefore, as the result is true for $n=1$ and if its true for $n=k$, its true for $n=k+1$ so by mathematical induction, this result is true for all positive integers $\geq \underline{1}$

6 A linear transformation T of the x-y plane has an associated matrix **M**, where $\mathbf{M} = \begin{pmatrix} \lambda & k \\ 1 & \lambda - k \end{pmatrix}$, and λ and k are real constants.

(a) You are given that $\det \mathbf{M} > 0$ for all values of λ.

(i) Find the range of possible values of k. **[3]**

(ii) What is the significance of the condition $\det \mathbf{M} > 0$ for the transformation T? **[1]**

For the remainder of this question, take $k = -2$.

(b) Determine whether there are any lines through the origin that are invariant lines for the transformation T. **[4]**

(c) The transformation T is applied to a triangle with area 3 units^2. The area of the resulting image triangle is 15 units^2.
Find the possible values of λ. **[3]**

a) i) $\det M = \lambda(\lambda - k) - k = \lambda^2 - k\lambda - k$
$\det M > 0$ so $\lambda^2 - k\lambda - k > 0$. This means no real roots so discriminant of $\lambda^2 - k\lambda - k$ is < 0 so $k^2 - 4k < 0$
$k(k-4) < 0 \implies -4 < k < 0$

ii) A positive determinant means the transformation preserves the orientation of the shapes.

b) $\begin{pmatrix} \lambda & -2 \\ 1 & \lambda + 2 \end{pmatrix} \begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} \lambda x - 2mx \\ x + (m\lambda + 2m)x \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

don't need $+c$ as through origin

As $y = mx$: $x + (m\lambda + 2m)x = m(\lambda x - 2mx)$
so $(2m + m\lambda + 1)x = (-2m^2 + m\lambda)x$
$0 = (-2m^2 - 2m - 1)x$. As $-2m^2 - 2m - 1$ has no real solutions, there are no invariant lines through origin

c) $\det M = \frac{15}{3} = 5 = \lambda^2 + 2\lambda + 2$

$\Rightarrow \lambda^2 + 2\lambda - 3 = 0$ so $(\lambda+3)(\lambda-1)=0$

$\therefore \lambda = -3$ or 1

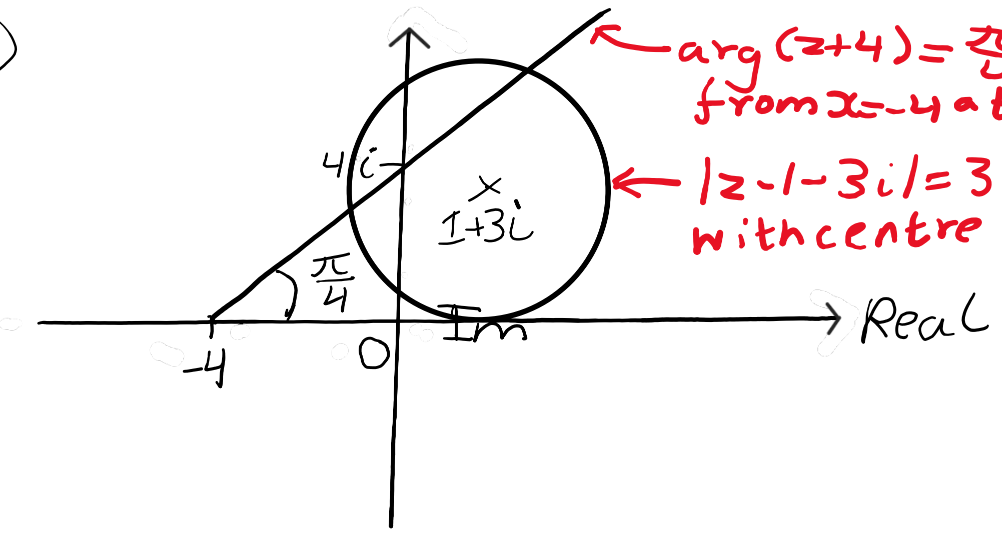
7 **(a)** Sketch on a single Argand diagram

(i) the set of points for which $|z-1-3\mathrm{i}| = 3$, **[3]**

(ii) the set of points for which $\arg(z+4) = \frac{1}{4}\pi$. **[3]**

(b) Find, in exact form, the two values of z for which $|z-1-3\mathrm{i}| = 3$ and $\arg(z+4) = \frac{1}{4}\pi$. **[6]**

a)



$\arg(z+4) = \frac{\pi}{4}$ is a half line from $x=-4$ at angle of $\frac{\pi}{4}$

$|z-1-3i| = 3$ is a circle with centre $1+3i$ and radius 3

b) $\arg(z+4) = \frac{\pi}{4}$ can be converted to $y = x+4$ in cartesian form and $|z-1-3i| = 3$ can be converted to $(x-1)^2 + (y-3)^2 = 9$ using centre $(1,3)$ and radius $= 3$.

Substitute $y = x+4$ into $(x-1)^2 + (y-3)^2 = 9$:

$(x-1)^2 + (x+1)^2 = 9$ so $(x^2 - 2x + 1) + (x^2 + 2x + 1) = 9$

$\Rightarrow 2x^2 + 2 = 9$ so $2x^2 = 7$, $x^2 = \frac{7}{2}$ so $x = \pm\sqrt{\frac{7}{2}}$

As $y = x+4$, $y = 4 \pm \sqrt{\frac{7}{2}}$

Therefore, $z = \sqrt{\frac{7}{2}} + \left(4 + \sqrt{\frac{7}{2}}\right)i$ or $-\sqrt{\frac{7}{2}} + \left(4 - \sqrt{\frac{7}{2}}\right)i$

8 In this question you must show detailed reasoning.

You are given that i is a root of the equation $z^4 - 2z^3 + 3z^2 + az + b = 0$, where a and b are real constants.

(a) Show that $a = -2$ and $b = 2$. **[4]**

(b) Find the other roots of this equation. **[7]**

a) Substitute $z = i$ into equation as it is a root:

$i^4 - 2i^3 + 3i^2 + ai + b = 0$ so $1 + 2i - 3 + ai + b = 0$.

$\Rightarrow (2+a)i + (b-2) = 0$

Equate imaginary: $2 + a = 0 \Rightarrow a = -2$

Equate real: $b - 2 = 0 \Rightarrow b = 2$

b) As one root is i, another root must be $-i$ because if a complex number is a root, its conjucate is also a root so $\alpha = i$ and $\beta = -i$. $(z-i)(z+i) = z^2 + 1$

$z^4 - 2z^3 + 3z^2 - 2z + 2 = (z^2+1)(z^2 + cz + d)$
$= z^4 + cz^3 + (d+1)z^2 + cz + d$

By equating them: $c = -2$ and $d = 2$ so $z^2 - 2z + 2$

$\Rightarrow z = \frac{2 \pm \sqrt{4 - 8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$

Therefore, the roots are $\alpha = i$, $\beta = -i$, $\gamma = 1 + i$, $\delta = 1 - i$.

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