OCR
Oxford Cambridge and RSA

AS Level Further Mathematics B (MEI)

Y410/01 Core Pure
Question Paper

Monday 14 May 2018 – Afternoon

Time allowed: 1 hour 15 minutes

You must have:
- Printed Answer Booklet
- Formulae Further Mathematics B (MEI)

You may use:
- a scientific or graphical calculator

Model Answers

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INSTRUCTIONS
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Complete the boxes provided on the Printed Answer Booklet with your name, centre number and candidate number.
- Answer **all** the questions.
- **Write your answer to each question in the space provided in the Printed Answer Booklet.** Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- Do **not** write in the barcodes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION
- The total number of marks for this paper is **60**.
- The marks for each question are shown in brackets **[ ]**.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is used. You should communicate your method with correct reasoning.
- The Printed Answer Booklet consists of **12** pages. The Question Paper consists of **4** pages.

Turn over

Answer **all** the questions.

1 The matrices **A**, **B** and **C** are defined as follows:

$$\mathbf{A} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 2 & 0 & 3 \\ 1 & -1 & 3 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 1 & 3 \end{pmatrix}.$$

Calculate all possible products formed from two of these three matrices. **[4]**

$$AC = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\begin{pmatrix} 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 6 \\ 3 & 9 \end{pmatrix}$$

$$BA = \begin{pmatrix} 2 & 0 & 3 \\ 1 & -1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 11 \\ 8 \end{pmatrix}$$

$$CB = \begin{pmatrix} 1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 0 & 3 \\ 1 & -1 & 3 \end{pmatrix} = \begin{pmatrix} 5 & -3 & 12 \end{pmatrix}$$

2 Find, to the nearest degree, the angle between the vectors $\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 3 \\ -3 \end{pmatrix}$. **[3]**

$a = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ and $b = \begin{pmatrix} -2 \\ 3 \\ -3 \end{pmatrix}$ $\quad |a| = \sqrt{1^2 + 0^2 + (-2)^2} = \sqrt{5}$

$|b| = \sqrt{(-2)^2 + 3^2 + (-3)^2} = \sqrt{22}$

$a \cdot b = (1 \times -2) + (0 \times 3) + (-2 \times -3) = -2 + 6 = 4$

$\cos\theta = \dfrac{a \cdot b}{|a||b|} = \dfrac{4}{\sqrt{5}\sqrt{22}} \Rightarrow \theta = 67.58 = 68^\circ$ to nearest degree

3 Find real numbers a and b such that $(a-3\mathrm{i})(5-\mathrm{i}) = b - 17\mathrm{i}$. **[5]**

$(a-3i)(5-i) = 5a - 15i - ai + 3i^2$

$= 5a - 3 - (15+a)i$

$\therefore 5a - 3 - (15+a)i = b - 17i$

Equate Imaginary: $15 + a = 17 \Rightarrow a = 2$

Equate Real: $5a + 3 = b$ so $5(2) - 3 = b$

$\Rightarrow b = 7$

4 Find a cubic equation with real coefficients, two of whose roots are $2 - \mathrm{i}$ and 3. **[5]**

If $2-i$ is one of the roots then $2+i$ is also a root as if a complex number is a root, its conjucate is also a root.

$\alpha = 3$, $\beta = 2+i$ and $\gamma = 2-i$

$\Sigma\alpha = 3+2+i+2-i = 7 = -\frac{b}{a}$ so assuming $a=1 \Rightarrow b=-7$

$\Sigma\alpha\beta = 3(2+i)+3(2-i)+(2+i)(2-i)$
$= 6+3i+6-3i+4+2i-2i-i^2$
$= 17 = \frac{c}{a} \Rightarrow c = 17$

$\alpha\beta\gamma = 3(2+i)(2-i) = 3\times 5 = 15 = -\frac{d}{a} \Rightarrow d = -15$

$\therefore$ equation is $z^3 - 7z^2 + 17z - 15 = 0$

5 A transformation of the x-y plane is represented by the matrix $\begin{pmatrix} \cos\theta & 2\sin\theta \\ 2\sin\theta & -\cos\theta \end{pmatrix}$, where θ is a positive acute angle.

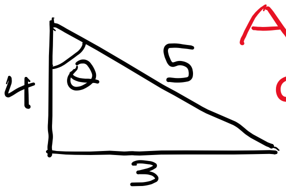
(i) Write down the image of the point (2, 3) under this transformation. **[2]**

$$\begin{pmatrix} \cos\theta & 2\sin\theta \\ 2\sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2\cos\theta + 6\sin\theta \\ 4\sin\theta - 3\cos\theta \end{pmatrix}$$

(ii) You are given that this image is the point $(a, 0)$. Find the value of a. **[5]**

$2\cos\theta + 6\sin\theta = a$ and $4\sin\theta - 3\cos\theta = 0$

$\therefore 4\sin\theta = 3\cos\theta$ so $\frac{\sin\theta}{\cos\theta} = \frac{3}{4}$. $\tan\theta = \frac{3}{4}$



As $\tan\theta = \frac{3}{4}$ you can draw a right angled triangle with sides 3 and 4 and by Pythagoras, hypotenuse = 5

Thus, from the triangle: $\sin\theta = \frac{O}{H} = \frac{3}{5}$ and $\cos\theta = \frac{4}{5}$.

$2\cos\theta + 6\sin\theta = 2\left(\frac{4}{5}\right) + 6\left(\frac{3}{5}\right) = 5.2$. $\therefore$ $\underline{a = 5.2}$

6 Find the invariant line of the transformation of the x-y plane represented by the matrix $\begin{pmatrix} 2 & 0 \\ 4 & -1 \end{pmatrix}$. **[4]**

$$\begin{pmatrix} 2 & 0 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x \\ mx+c \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Invariant line is a line that remains unchanged by the transformation

$$\begin{pmatrix} 2x \\ 4x-mx-c \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Substitute these x and y values into $y = mx+c$

$4x - mx - c = m(2x) + c$ so $(4-m)x = 2mx$

$\Rightarrow 4 - m = 2m$ so $4 = 3m$ and $m = \frac{4}{3}$.

As $m = \frac{4}{3}$, $c = 0$; Invariant Line is $y = \frac{4}{3}x$

7 **(i)** Express $\dfrac{1}{2r-1} - \dfrac{1}{2r+1}$ as a single fraction. **[2]**

$$\frac{1}{2r-1} - \frac{1}{2r+1} = \frac{2r+1-(2r-1)}{(2r+1)(2r-1)} = \frac{2}{(2r+1)(2r-1)}$$

(ii) Find how many terms of the series

$$\frac{2}{1\times3} + \frac{2}{3\times5} + \frac{2}{5\times7} + \ldots + \frac{2}{(2r-1)(2r+1)} + \ldots$$

are needed for the sum to exceed 0.999 999. **[7]**

$$\frac{2}{1\times3} + \frac{2}{3\times5} + \frac{2}{5\times7} + \ldots + \frac{2}{(2n-1)(2n+1)}$$

$$= \sum_{r=1}^{n} \frac{2}{(2r-1)(2r+1)} = \sum_{r=1}^{n}\left(\frac{1}{2r-1} - \frac{1}{2r+1}\right)$$

$= 1 - \frac{1}{3}$

$+ \frac{1}{3} - \frac{1}{5}$

$+ \frac{1}{5} - \frac{1}{7}$

$+ \ldots\ldots$

$+ \frac{1}{2n-1} - \frac{1}{2n+1}$

Substitute $r = 1$ then $r = 2, 3$ etc. up to $r = n$ and see what cancels out

All cancels out

$\therefore \sum_{r=1}^{n}\left(\frac{1}{2r-1}+\frac{1}{2r+1}\right)=1-\frac{1}{2n+1}$.

$1-\frac{1}{2n+1}>0.999999$ so $0.000001>\frac{1}{2n+1}$

$2n+1>1,000,000 \Rightarrow n>499,999.5$

Thus number of terms $=500,000$

8 Prove by induction that $\begin{pmatrix}1 & 1\\0 & 2\end{pmatrix}^n=\begin{pmatrix}1 & 2^n-1\\0 & 2^n\end{pmatrix}$ for all positive integers n. **[6]**

When $n=1$, LHS $=\begin{pmatrix}1 & 1\\0 & 2\end{pmatrix}^1=\begin{pmatrix}1 & 1\\0 & 2\end{pmatrix}$.

RHS $=\begin{pmatrix}1 & 2-1\\0 & 2\end{pmatrix}=\begin{pmatrix}1 & 1\\0 & 2\end{pmatrix}$. LHS=RHS so true for $n=1$

Assume it is true for $n=k$ so

$\begin{pmatrix}1 & 1\\0 & 2\end{pmatrix}^k=\begin{pmatrix}1 & 2^k-1\\0 & 2^k\end{pmatrix}$

For $n=k+1$: $\begin{pmatrix}1 & 1\\0 & 2\end{pmatrix}^{k+1}=\begin{pmatrix}1 & 1\\0 & 2\end{pmatrix}^k\begin{pmatrix}1 & 1\\0 & 2\end{pmatrix}$

$=\begin{pmatrix}1 & 2^k-1\\0 & 2^k\end{pmatrix}\begin{pmatrix}1 & 1\\0 & 2\end{pmatrix}=\begin{pmatrix}1 & 1+2^{k+1}-2\\0 & 2^{k+1}\end{pmatrix}$

$\begin{pmatrix}1 & 2^{k+1}-1\\0 & 2^{k+1}\end{pmatrix}$

Therefore, if it is true for $n=k$, its true for $n=k+1$. As it is true for $n=1$, it is true for all positive integers by mathematical induction.

9 Fig. 9 shows a sketch of the region OPQ of the Argand diagram defined by

$$\left\{z : |z| \leqslant 4\sqrt{2}\right\} \cap \left\{z : \tfrac{1}{4}\pi \leqslant \arg z \leqslant \tfrac{1}{3}\pi\right\}.$$

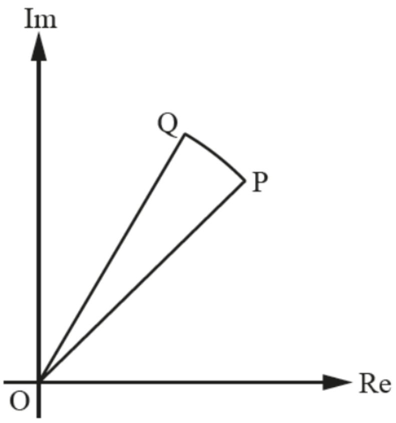


Fig. 9

(i) Find, in modulus-argument form, the complex number represented by the point P. **[2]**

At point P, $|z| = 4\sqrt{2}$ and $\arg z = \frac{\pi}{4}$ so $P = 4\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

(ii) Find, in the form $a + ib$, where a and b are exact real numbers, the complex number represented by the point Q. **[3]**

At Q, $|z| = 4\sqrt{2}$ and $\arg z = \frac{\pi}{3}$
so $Q = 4\sqrt{2}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$
$= 2\sqrt{2} + 2\sqrt{6}\,i$

(iii) **In this question you must show detailed reasoning.**

Determine whether the points representing the complex numbers

- $3 + 5i$
- $5.5(\cos 0.8 + i\sin 0.8)$

lie within this region. **[4]**

$|3+5i| = \sqrt{3^2+5^2} = \sqrt{34} > \sqrt{32}$
so $3+5i$ doesn't lie in the region.
$5.5(\cos 0.8 + i\sin 0.8)$:
$5.5 < 4\sqrt{2}$ and $\frac{\pi}{4} < 0.8 < \frac{\pi}{3}$
so this point does lie in the region

10 Three planes have equations

$$-x + 2y + z = 0$$
$$2x - y - z = 0$$
$$x + y = a$$

where a is a constant.

(i) Investigate the arrangement of the planes:

- when $a = 0$;
- when $a \neq 0$. **[6]**

$$\begin{pmatrix} -1 & 2 & 1 \\ 2 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix}.$$

Let $M = \begin{pmatrix} -1 & 2 & 1 \\ 2 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix} \Rightarrow \det M = 0$ so no unique solution

When $a = 0$;

(3) $x + y = 0 \Rightarrow x = -y$

(1) $-x + 2y + z = 0 \Rightarrow 3y + z = 0$

(2) $2x - y - z = 0 \Rightarrow 3y + z = 0$

Thus when $a = 0$, the planes intersect in a line forming a sheaf.

When $a \neq 0$, there are no solutions so the planes form a prism formation as all 3 planes never intersect.

(ii) Chris claims that the position vectors $-\mathbf{i}+2\mathbf{j}+\mathbf{k}$, $2\mathbf{i}-\mathbf{j}-\mathbf{k}$ and $\mathbf{i}+\mathbf{j}$ lie in a plane. Determine whether or not Chris is correct. **[2]**

These are the normals to the 3 planes so regardless of the value of a, they must all lie in the same plane

END OF QUESTION PAPER