OCR
Oxford Cambridge and RSA

Thursday 13 June 2019 – Afternoon

A Level Further Mathematics B (MEI)

Y421/01 Mechanics Major

Time allowed: 2 hours 15 minutes

You must have:
- Printed Answer Booklet
- Formulae Further Mathematics B (MEI)

You may use:
- a scientific or graphical calculator

MODEL SOLUTIONS

INSTRUCTIONS
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Answer **all** the questions.
- **Write your answer to each question in the space provided in the Printed Answer Booklet.** If additional space is required, you should use the lined page(s) at the end of the Printed Answer Booklet. The question number(s) must be clearly shown.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The acceleration due to gravity is denoted by $g\,\mathrm{m\,s^{-2}}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION
- The total mark for this paper is **120**.
- The marks for each question are shown in brackets **[ ]**.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is used. You should communicate your method with correct reasoning.
- The Printed Answer Booklet consists of **24** pages. The Question Paper consists of **12** pages.

1 Three forces represented by the vectors $-4\mathbf{i}$, $\mathbf{i}+2\mathbf{j}$ and $k\mathbf{i}-2\mathbf{j}$ act at the points with coordinates (0, 0), (3, 0) and (0, 4) respectively.

(a) Given that the three forces form a couple, find the value of k. **[2]**

(b) Find the magnitude and direction of the couple. **[3]**

a. Resolve in i:

$-4 + 1 + k = 0$

$k = 3$

b. Take moments about O:

$4(k) - 3(2)$

$= 12 - 6$

$= 6$ Nm clockwise

2 The Reynolds number, R, is an important dimensionless quantity in fluid dynamics; it can be used to predict flow patterns when a fluid is in motion relative to a surface.

The Reynolds number is defined as $R = \dfrac{\rho u l}{\mu}$,

where ρ is the density of the fluid, u is the velocity of the fluid relative to the surface, l is the distance travelled by the fluid and μ is the viscosity of the fluid.

Find the dimensions of μ. **[4]**

$[\rho] = ML^{-3}$

$[u] = LT^{-1}$

$[l] = L$

$\therefore [\mu] = [\rho u l] = ML^{-3}LT^{-1}L = ML^{-1}T^{-1}$

3 A ball of mass 2 kg is moving with velocity $(3\mathbf{i}-2\mathbf{j})\,\mathrm{m\,s^{-1}}$ when it is struck by a bat. The impulse on the ball is $(-8\mathbf{i}+10\mathbf{j})\,\mathrm{N\,s}$.

(a) Find the speed of the ball immediately after the impact. **[4]**

(b) State one modelling assumption you have used in answering part **(a)**. **[1]**

a. $I = \Delta mv$

$$-8\underline{i} + 10\underline{j} = 2(\underline{v} - (3\underline{i} - 2\underline{j}))$$

$$\therefore \underline{v} = (-4+3)\underline{i} + (5-2)\underline{j}$$

$$\underline{v} = -\underline{i} + 3\underline{j}$$

$$|\underline{v}| = \sqrt{1^2+3^2} = \sqrt{10}\ \mathrm{ms^{-1}}$$

b. The ball is modelled as a particle.

4

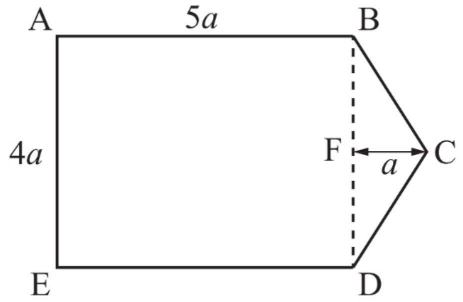


Fig. 4

Fig. 4 shows a uniform lamina ABCDE such that ABDE is a rectangle and BCD is an isosceles triangle. AB $= 5a$, AE $= 4a$ and BC = CD. The point F is the midpoint of BD and FC $= a$.

(a) Find, in terms of a, the exact distance of the centre of mass of the lamina from AE. **[4]**

The lamina is freely suspended from B and hangs in equilibrium.

(b) Find the angle between AB and the downward vertical. **[2]**

a. $\bar{x}\left((4a \times 5a) + \frac{1}{2} \times 4a \times a\right) = \left(\frac{5}{2}a\right)\left(4a \times 5a\right) + \left(5a + \frac{1}{3}a\right)\left(\frac{1}{2}(4a)(a)\right)$

$$\Rightarrow \bar{x} = \frac{50a^3 + \frac{32}{3}a^3}{22a^2} = \frac{91}{33}a$$

b. $\tan\theta = \frac{opp}{adj} = \frac{2a}{5a - \bar{x}} = \frac{2a}{5a - \frac{91}{33}a} = \frac{33}{37}$

$\therefore \theta = \tan^{-1}\left(\frac{33}{37}\right) = 41.7^\circ$

5 A particle P of mass 4 kilograms moves in such a way that its position vector at time t seconds is $\mathbf{r}$ metres, where

$$\mathbf{r} = 3t\mathbf{i} + 2e^{-3t}\mathbf{j}\ .$$

(a) Find the initial kinetic energy of P. **[4]**

(b) Find the time when the acceleration of P is 2 metres per second squared. **[3]**

a. $\frac{dr}{dt} = \underline{v} = 3\underline{i} - 6e^{-3t}\underline{j}$

initially, $t = 0$, $\underline{v} = 3\underline{i} - 6\underline{j}$

$|\underline{v}| = \sqrt{3^2+6^2} = \sqrt{45}$

$\therefore$ initial $KE = \frac{1}{2} \times 4 \times (\sqrt{45})^2$

$= 90\ J$

b. $\underline{a} = \frac{d^2r}{dt^2} = 0\underline{i} + 18e^{-3t}\underline{j}$

at $\underline{a} = 2$, $18e^{-3t} = 2$

$-3t = \ln\frac{1}{9}$

$t = \frac{1}{3}\ln 9$

$t = 0.732\ s$

6

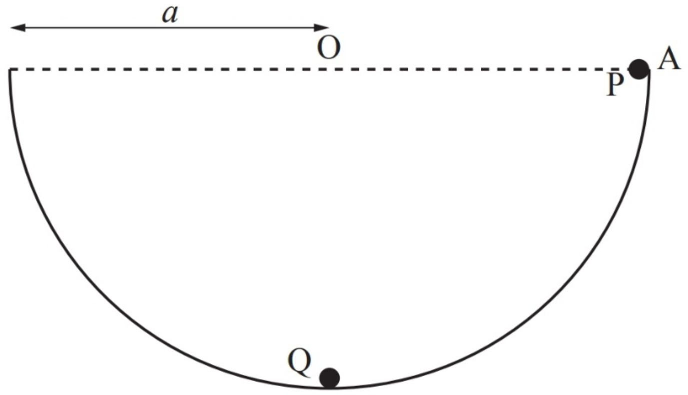


Fig. 6

The rim of a smooth hemispherical bowl is a circle of centre O and radius a. The bowl is fixed with its rim horizontal and uppermost. A particle P of mass m is released from rest at a point A on the rim as shown in Fig. 6.

When P reaches the lowest point of the bowl it collides directly with a stationary particle Q of mass $\frac{1}{2}m$. After the collision Q just reaches the rim of the bowl.

Find the coefficient of restitution between P and Q. **[7]**

KE gained by P = PE lost by P

$$\frac{1}{2}mv^2 = mga$$

$$\therefore v^2 = 2ga$$

$$v = \sqrt{2ga}$$

Conservation of momentum:

$$m\sqrt{2ga} + \frac{1}{2}m \times 0 = mv_P + \frac{1}{2}mv_Q$$

$$\Rightarrow v_P + \frac{1}{2}v_Q = \sqrt{2ga}$$

$$v_P = \sqrt{2ga} - \frac{1}{2}v_Q \qquad (1)$$

Newton's law of restitution:

$$e = \frac{V_Q - V_P}{\sqrt{2ga} - 0}$$

(sub in ①)

$$e\sqrt{2ga} = V_Q - \left(\sqrt{2ga} - \tfrac{1}{2}V_Q\right)$$

$$e\sqrt{2ga} = \tfrac{3}{2}V_Q - \sqrt{2ga}$$

$$\tfrac{3}{2}V_Q = \sqrt{2ga} + e\sqrt{2ga}$$

$$V_Q = \tfrac{2}{3}\sqrt{2ga}\,(1+e)$$

We know that initial KE of Q after collision $= \frac{1}{2}mga$ as it <u>just</u> reaches the rim.

$$\therefore \tfrac{1}{2}\left(\tfrac{1}{2}m\right)V_Q^2 = \tfrac{1}{2}mga$$

$$\tfrac{1}{4}m\left(\tfrac{2}{3}\sqrt{2ga}(1+e)\right)^2 = \tfrac{1}{2}mga$$

$$\tfrac{4}{9}\times 2ga(1+e^2+2e) = \frac{\frac{1}{2}mga}{\frac{1}{4}m}$$

$$e^2+2e+1 = \frac{2ga}{\frac{4}{9}\times 2ga}$$

$$e^2+2e-\tfrac{5}{4} = 0$$

By calculator, $e = \frac{1}{2}$ or $e = -\frac{5}{2}$ ← $-1 \le e \le 1$ ∴ disregard

7 In this question you must show detailed reasoning.

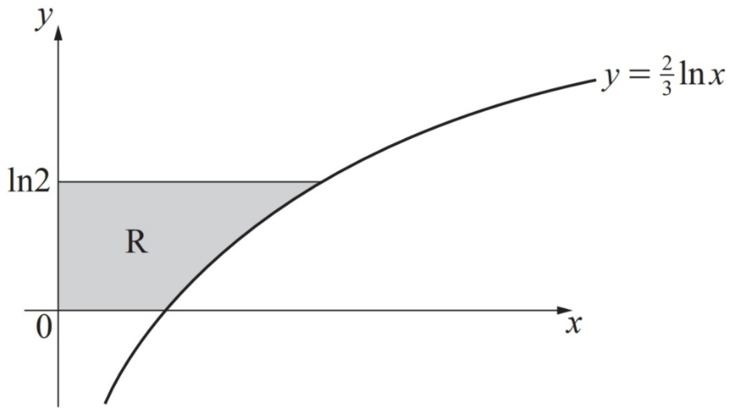


Fig. 7

Fig. 7 shows the curve with equation $y = \frac{2}{3}\ln x$. The region R, shown shaded in Fig. 7, is bounded by the curve and the lines $x = 0$, $y = 0$ and $y = \ln 2$. A uniform solid of revolution is formed by rotating the region R completely about the y-axis.

Find the exact y-coordinate of the centre of mass of the solid. **[8]**

$$y = \frac{2}{3}\ln x$$

$$\Rightarrow x = e^{\frac{3}{2}y}$$

$$V = \pi\int_a^b x^2\,dy$$

$$= \pi\int_0^{\ln 2}\left(e^{\frac{3}{2}y}\right)^2 dy$$

$$= \pi\left[\frac{1}{3}e^{3y}\right]_0^{\ln 2}$$

$$= \pi\left(\frac{2^3}{3} - \frac{1}{3}\right)$$

$$= \frac{7}{3}\pi$$

$$\bar{y} = \frac{\pi}{V} \int y x^2 \, dy$$

$$\therefore V\bar{y} = \pi \int_0^{\ln 2} \left(y e^{3y} \right) dy$$

$$= \pi \left(\left[\frac{1}{3} y e^{3y} \right]_0^{\ln 2} - \frac{1}{3} \int_0^{\ln 2} e^{3y} \, dy \right)$$

$$= \pi \left[\frac{1}{3} y e^{3y} - \frac{1}{9} e^{3y} \right]_0^{\ln 2}$$

$$\frac{7}{3} \cancel{\pi} \bar{y} = \cancel{\pi} \cdot \frac{8}{3} \ln 2 - \frac{7}{9}$$

$$\bar{y} = \frac{8}{7} \ln 2 - \frac{1}{3}$$

8 A car of mass 800 kg travels up a line of greatest slope of a straight road inclined at 5° to the horizontal.

The power developed by the car is constant and equal to 25 kW. The resistance to the motion of the car is constant and equal to 750 N.

The car passes through a point A on the road with speed $7\,\text{m s}^{-1}$.

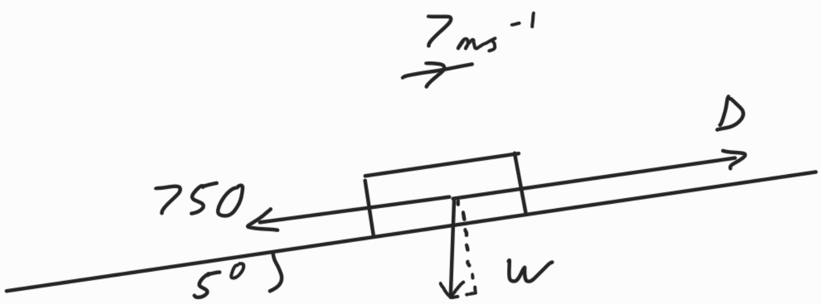
(a) Find

- the acceleration of the car at A,
- the greatest steady speed at which the car can travel up the hill. **[5]**

The car later passes through a point B on the road where $AB = 131\,\text{m}$. The time taken to travel from A to B is 10.4 s.

(b) Calculate the speed of the car at B. **[6]**

a.



$$f = \frac{P}{v} \Rightarrow D = \frac{25 \times 10^3}{7}\ \text{N}$$

Σf up the slope :

$$D - 750 - W \sin 5$$

$$= \frac{25000}{7} - 750 - 800g \sin 5$$

By Newton II:

$$\frac{25000}{7} - 750 - 800g\sin 5 = 800a$$

$$\Rightarrow a = 2.67\ ms^{-2}$$

By setting $a = 0$, and $v = v$, we can find the greatest steady speed:

$$\frac{25000}{v} - 750 - 800g\sin 5 = 800 \times 0$$

$$\Rightarrow v = 17.4\ ms^{-1}$$

b. WD by car $= Pt$
$= 25000 \times 10.4$
$= 260000$

WD against friction $= Fd$
$= 750 \times 131$
$= 98250$

Change in PE $= 800g \times 131\sin 5$

Change in KE $= \frac{800}{2}(v^2 - 7^2)$

Change in energy = work done :

$$400(v^2 - 7^2) + 800g(131 \sin 5) = 260000 - 98250$$

$$400(v^2 - 7^2) = 72237.566$$

$$v^2 = 229.594$$

$$v = 15.2 \text{ ms}^{-1}$$

9

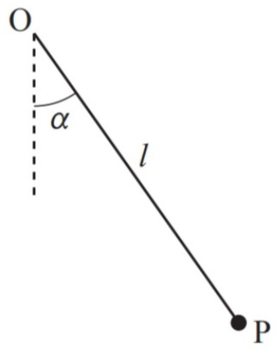


Fig. 9

A particle P of mass m is joined to a fixed point O by a light inextensible string of length l. P is released from rest with the string taut and making an acute angle α with the downward vertical, as shown in Fig. 9.

At a time t after P is released the string makes an angle θ with the downward vertical and the tension in the string is T. Angles α and θ are measured in radians.

(a) Show that

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{2g}{l}\cos\theta + k_1,$$

where k_1 is a constant to be determined in terms of g, l and α. **[4]**

(b) Show that

$$T = 3mg\cos\theta + k_2,$$

where k_2 is a constant to be determined in terms of m, g and α. **[3]**

It is given that α is small enough for α^2 to be negligible.

(c) Find, in terms of m and g, the approximate tension in the string. **[2]**

(d) Show that the motion of P is approximately simple harmonic. **[3]**

a. Initial $PE = -mgl\cos\alpha$
Initial $KE = 0$

(using O as point of 0 GPE)

at θ: $PE = -mgl\cos\theta$
$KE = \frac{1}{2}mv^2$

Conservation of energy and sub in $v = l\frac{d\theta}{dt}$:

$$\frac{1}{2}\cancel{m}\left(l\frac{d\theta}{dt}\right)^2 + (-\cancel{m}g\, l\cos\theta) = 0 + (-\cancel{m}g\, l\cos\alpha)$$

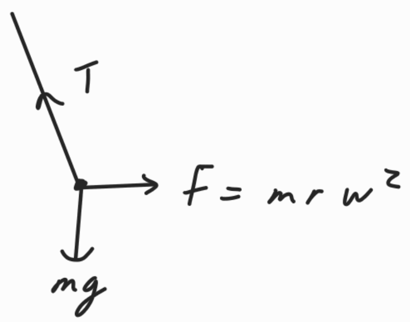
$$l^2\left(\frac{d\theta}{dt}\right)^2 = 2gl(\cos\theta - \cos\alpha)$$

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{2g}{l}\cos\theta - \frac{2g}{l}\cos\alpha$$ as required

where

$\Rightarrow k_1 = -\frac{2g}{l}\cos\alpha$

b.



By Newton II :

$$T - mg\cos\theta = ml\left(\frac{2g}{l}(\cos\theta - \cos\alpha)\right)$$

$$T = 3mg\cos\theta - 2mg\cos\alpha$$

$\therefore k_2 = -2mg\cos\alpha$

c. $\alpha^2 \approx 0 \Rightarrow \cos\alpha \approx 1$

$\therefore \cos\theta \approx 1$

$\therefore T \approx 3mg - 2mg$

$T \approx mg$

d. differentiate $\left(\frac{d\theta}{dt}\right)^2 = \frac{2g}{l}\cos\theta - \frac{2g}{l}\cos\alpha$ w.r.t. t:

$(\cos\alpha \approx 1)$

$2\cancel{\left(\frac{d\theta}{dt}\right)}\left(\frac{d^2\theta}{dt^2}\right) = \frac{2g}{l}(-\sin\theta)\cancel{\left(\frac{d\theta}{dt}\right)}$

$(\sin\theta \approx \theta$ for small $\theta)$

$2\frac{d^2\theta}{dt^2} \approx -\frac{2g\theta}{l}$

$\left(w^2 = \frac{g}{l}\right)$

$\frac{d^2\theta}{dt^2} \approx -w^2\theta$

So motion is approximately simple harmonic.

10 A particle P, of mass m, moves on a rough horizontal table. P is attracted towards a fixed point O on the table by a force of magnitude $\dfrac{kmg}{x^2}$, where x is the distance OP.

The coefficient of friction between P and the table is μ.

P is initially projected in a direction directly away from O. The velocity of P is first zero at a point A which is a distance a from O.

(a) Show that the velocity v of P, when P is moving away from O, satisfies the differential equation

$$\frac{d}{dx}(v^2)+\frac{2kg}{x^2}+2\mu g=0.$$

[3]

(b) Verify that

$$v^2 = 2gk\left(\frac{1}{x}-\frac{1}{a}\right)+2\mu g(a-x).$$

[3]

(c) Find, in terms of k and a, the range of values of μ for which P remains at A. **[2]**

a. $F = ma$:

$$-\frac{kmg}{x^2} - f = ma \qquad (f = \mu R = \mu mg)$$

$$-\frac{kmg}{x^2} - \mu mg = ma \qquad \left(a = v\frac{dv}{dx}\right)$$

$$-\frac{kg}{x^2} - \mu g = v\frac{dv}{dx}$$

$$v\frac{dv}{dx} + \frac{kg}{x^2} + \mu g = 0 \qquad \left(v\frac{dv}{dx} = \frac{1}{2}\frac{d}{dx}(v^2)\right)$$

$$\frac{d}{dx}(v^2) + \frac{2kg}{x^2} + 2\mu g = 0$$

as required

b. When $x = a$: $v^2 = 2gk\left(\frac{1}{a} - \frac{1}{a}\right) + 2\mu g(a-a) = 0$

$\therefore v = 0$ at $x = a$

$$\frac{d}{dx}(v^2) = \frac{d}{dx}\left(2gk\left(\frac{1}{x} - \frac{1}{a}\right) + 2\mu g(a-x)\right)$$

(a and $\frac{1}{a}$ terms are just constants, so these are lost)

$$\frac{d(v^2)}{dx} = -\frac{2gk}{x^2} - 2\mu g \quad ①$$

From (a): $\frac{d}{dx}(v^2) + \frac{2kg}{x^2} + 2\mu g = 0$

Sub in ①: $-\frac{2gk}{x^2} - 2\mu g + \frac{2kg}{x^2} + 2\mu g = 0$

This is consistent $\therefore$ equation is verified.

c. Remains at A if friction $\geq$ attraction force

$$\mu mg \geq \frac{kmg}{a^2}$$

$$\mu \geq \frac{k}{a^2}$$

11 Two uniform smooth spheres A and B have equal radii and are moving on a smooth horizontal surface. The mass of A is 0.2 kg and the mass of B is 0.6 kg.

The spheres collide obliquely. When the spheres collide the line joining their centres is parallel to **i**.

Immediately before the collision the velocity of A is $\mathbf{u}_A \,\mathrm{m\,s^{-1}}$ and the velocity of B is $\mathbf{u}_B \,\mathrm{m\,s^{-1}}$. The coefficient of restitution between A and B is 0.5.

Immediately after the collision the velocity of A is $(-4\mathbf{i}+2\mathbf{j})\,\mathrm{m\,s^{-1}}$ and the velocity of B is $(2\mathbf{i}+3\mathbf{j})\,\mathrm{m\,s^{-1}}$.

(a) Find $\mathbf{u}_A$ and $\mathbf{u}_B$. **[7]**

After the collision B collides with a smooth vertical wall which is parallel to **j**.

The loss in kinetic energy of B caused by the collision with the wall is 1.152 J.

(b) Find the coefficient of restitution between B and the wall. **[3]**

(c) Find the angle through which the direction of motion of B is deflected as a result of the collision with the wall. **[4]**

a. Let $\underline{u_A} = u_A \underline{i} + v_A \underline{j}$ and $\underline{u_B} = u_B \underline{i} + v_B \underline{j}$

Conservation of momentum:

$0.2 u_A + 0.6 \underline{u_B} = 0.2(-4) + 0.6(2)$

$u_A + 3u_B = 2 \Rightarrow u_A = 2 - 3u_B$ ①

Newton's Law of restitution:

$e = 0.5 = \dfrac{4+2}{u_A - u_B}$

$u_A - u_B = 12$ ②

Sub ① into ②: $2 - 3u_B - u_B = 12$

$\Rightarrow u_B = -2.5$

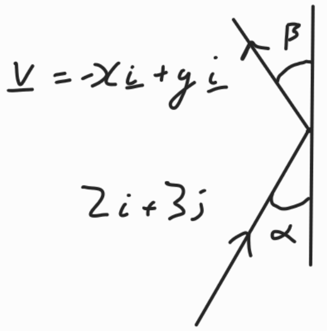
$\therefore\ u_A = 12 + (-2.5) = 9.5$

Oblique collision $\Rightarrow$ $\underline{j}$ components are constant

$\Rightarrow \underline{u_A} = 9.5\underline{i} + 2\underline{j}$

$\underline{u_B} = -2.5\underline{i} + 3\underline{j}$

b.



In $\underline{i}$ direction : $e = \dfrac{-x}{2}$

$\Rightarrow x = -2e$

$\underline{v} = -2e\underline{i} + 3\underline{j}$

$\Delta KE = \frac{1}{2} m \Delta v^2$

$1.152 = \frac{1}{2} \times 0.6 \times \left(\left(\sqrt{2^2+3^2}\right)^2 - \left(\sqrt{(-2e)^2+3^2}\right)^2 \right)$

$4e^2 + 9 = 9.16$

$e = \sqrt{0.04} = \underline{0.2}$

c. $\alpha = \tan^{-1}\left(\frac{2}{3}\right) = 33.690^\circ$

$\beta = \tan^{-1}\left(\frac{2 \times 0.2}{3}\right) = 7.595^\circ$

angle deflected through $= \alpha + \beta = 41.3^\circ$

12

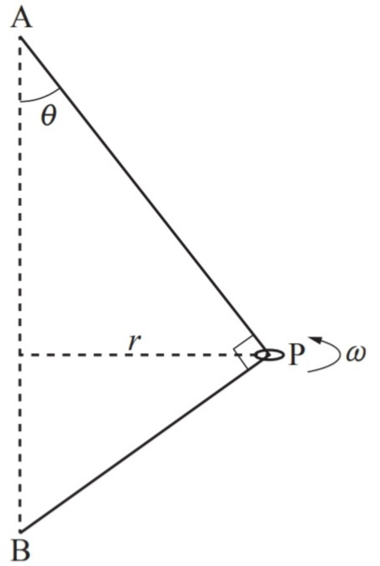


Fig. 12

The ends of a light inextensible string are fixed to two points A and B in the same vertical line, with A above B. The string passes through a small smooth ring of mass m. The ring is fastened to the string at a point P.

When the string is taut the angle APB is a right angle, the angle BAP is θ and the perpendicular distance of P from AB is r.

The ring moves in a horizontal circle with constant angular velocity ω and the string taut as shown in Fig. 12.

(a) By resolving horizontally and vertically, show that the tension in the part of the string BP is $m(r\omega^2\cos\theta - g\sin\theta)$. **[6]**

(b) Find a similar expression, in terms of r, ω, m, g and θ, for the tension in the part of the string AP. **[2]**

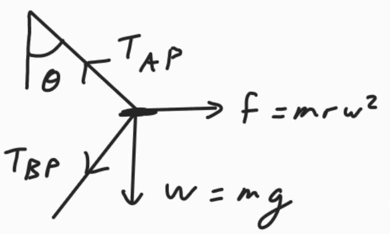
It is given that $AB = 5a$ and $AP = 4a$.

(c) Show that $16a\omega^2 > 5g$. **[3]**

The ring is now free to move on the string but remains in the same position on the string as before. The string remains taut and the ring continues to move in a horizontal circle.

(d) Find the period of the motion of the ring, giving your answer in terms of a, g and π. **[5]**

a.



Vertical resolving:

$T_{AP}\cos\theta = T_{BP}\sin\theta + mg$

Horizontal resolving:

$T_{AP}\sin\theta + T_{BP}\cos\theta = mrw^2$

Rearrange for T_{AP}:

$$T_{AP} = T_{BP}\tan\theta + mg\sec\theta$$

$$T_{AP} = \frac{mr\omega^2}{\sin\theta} - T_{BP}\cot\theta$$

$$\Rightarrow T_{BP}\tan\theta + mg\sec\theta = mr\omega^2\operatorname{cosec}\theta - T_{BP}\cot\theta$$

$(\times\cos\theta\times\sin\theta)$

$$T_{BP}\sin^2\theta + mg\sin\theta + T_{BP}\cos^2\theta = mr\omega^2\cos\theta$$

$$T_{BP}(\sin^2\theta + \cos^2\theta) = mr\omega^2\cos\theta - mg\sin\theta$$

$$T_{BP} = m(r\omega^2\cos\theta - g\sin\theta)$$

as required.

b. Sub equation for T_{BP} into an equation for T_{AP}:

$$T_{AP} = m(r\omega^2\cos\theta - g\sin\theta)\tan\theta + mg\sec\theta$$

$$= mr\omega^2\sin\theta - mg\frac{\sin^2\theta}{\cos\theta} + mg\times\frac{1}{\cos\theta}$$

$$= mr\omega^2\sin\theta + \frac{mg}{\cos\theta}(1-\sin^2\theta) \quad = \cos^2\theta$$

$$T_{AP} = m(r\omega^2\sin\theta + g\cos\theta)$$

c. $\Rightarrow \cos\theta = 4/5$, $\sin\theta = 3/5$

$$r = 4a\sin\theta = \frac{12}{5}a$$

Taut $\Rightarrow T_{BP} > 0$

$$\therefore \cancel{m} r w^2 \cos\theta - \cancel{m} g \sin\theta > 0$$

$$\frac{12a}{5} \times w^2 \times \frac{4}{5} > \frac{3}{5} g$$

$$48 a w^2 > 15 g$$

$$16 a w^2 > 5 g$$

d. Ring doesn't move $\Rightarrow$ $T_{AP} = T_{BP}$

$$\therefore \cancel{m}(r w^2 \cos\theta - g \sin\theta) = \cancel{m}(r w^2 \sin\theta + g\cos\theta)$$

$$\frac{12}{5} a \Omega^2 \times \frac{4}{5} - \frac{3}{5} g = \frac{12}{5} a \Omega^2 \times \frac{3}{5} + \frac{4}{5} g$$ (w is now Ω, angular speed)

$$\frac{12}{25} a \Omega^2 = \frac{7}{5} g$$

$$\Omega^2 = \frac{35 g}{12 a}$$

$$\Omega = \sqrt{\frac{35 g}{12 a}}$$

$$T = \frac{2\pi}{\Omega} = \frac{2\pi}{\sqrt{\frac{35g}{12a}}} = 2\pi\sqrt{\frac{12a}{35g}}$$

13

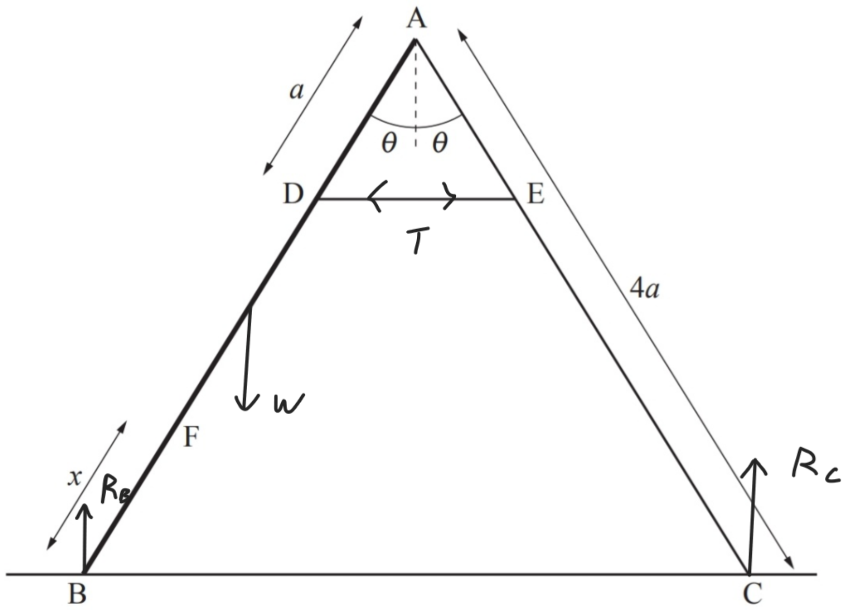


Fig. 13

A step-ladder has two sides AB and AC, each of length $4a$. Side AB has weight W and its centre of mass is at the half-way point; side AC is light.

The step-ladder is smoothly hinged at A and the two parts of the step-ladder, AB and AC, are connected by a light taut rope DE, where D is on AB, E is on AC and AD = AE = a.

A man of weight $4W$ stands at a point F on AB, where BF = x.

The system is in equilibrium with B and C on a smooth horizontal floor and the sides AB and AC are each at an angle θ to the vertical, as shown in Fig. 13.

(a) By taking moments about A for side AB of the step-ladder and then for side AC of the step-ladder show that the tension in the rope is

$$W\left(1+\frac{2x}{a}\right)\tan\theta.$$

[7]

The rope is elastic with natural length $\frac{1}{4}a$ and modulus of elasticity W.

(b) Show that the condition for equilibrium is that

$$x = \tfrac{1}{2}a(8\cos\theta - \cot\theta - 1).$$

[5]

In this question you must show detailed reasoning.

(c) Hence determine, in terms of a, the maximum value of x for which equilibrium is possible. **[5]**

a. Moments about A for AC :

$$4aR_C \sin\theta = aT\cos\theta \quad \Rightarrow \quad R_C = \frac{T\cos\theta}{4\sin\theta}$$

Moments about A for AB :

$$aT\cos\theta + 2aW\sin\theta + (4a - x)(4W\sin\theta) = 4aR_B\sin\theta$$

$$\Rightarrow R_B = \frac{aT\cos\theta + 2aW\sin\theta + 16aW\sin\theta - 4xW\sin\theta}{4a\sin\theta}$$

Resolve vertically :

$$R_B + R_C = 4W + W$$ (sub in equations for R_B and R_C)

$$\frac{aT\cos\theta + 18aW\sin\theta - 4xW\sin\theta}{4a\sin\theta} + \frac{T\cos\theta}{4\sin\theta} = 5W$$

$$T\cos\theta + 18W\sin\theta - \frac{4xW\sin\theta}{a} + T\cos\theta = 20W\sin\theta$$

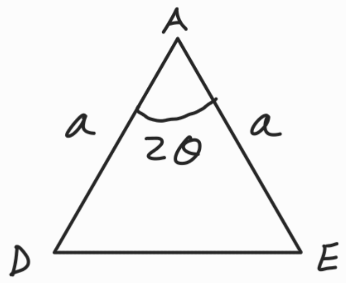
$$2aT\cos\theta - 2aW\sin\theta = 4xW\sin\theta$$

$$T = W\left(\frac{4x\sin\theta + 2a\sin\theta}{2a\cos\theta}\right)$$

$$T = W\left(\frac{2x\tan\theta}{a} + \tan\theta\right)$$

$$T = W\left(1 + \frac{2x}{a}\right)\tan\theta$$

b.



cosine rule:

$$DE^2 = a^2 + a^2 - 2a \times a \times \cos 2\theta$$

$$DE = \sqrt{2a^2 - 2a^2 \cos 2\theta}$$

Hooke's law: $T = \dfrac{\lambda x}{l}$ ← extension

$$T = \frac{W\left(DE - \frac{1}{4}a\right)}{\frac{1}{4}a}$$

(Sub in DE equation and T equation from a.)

$$W\left(1 + \frac{2x}{a}\right)\tan\theta = \frac{4W}{a}\left(\sqrt{2a^2 - 2a^2\cos 2\theta} - \frac{1}{4}a\right)$$

$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$\therefore 2a^2 - 2a^2\cos 2\theta = 2a^2 - (2a^2 + 4a^2\sin^2\theta)$$

$$\therefore \sqrt{2a^2 - 2a^2\cos 2\theta} = 2a\sin\theta$$

$$\Rightarrow \left(1 + \frac{2x}{a}\right)\tan\theta = \frac{4}{a}\left(2a\sin\theta - \frac{1}{4}a\right)$$

$$\frac{1}{4}a + \frac{1}{2}x = 2a\cos\theta - \frac{a}{4\tan\theta}$$

$$\frac{1}{2}x = a\left(2\cos\theta - \frac{1}{4}\cot\theta - \frac{1}{4}\right)$$

$$x = \frac{1}{2}a(8\cos\theta - \cot\theta - 1)$$

c. Maximum $\Rightarrow \frac{dx}{d\theta} = 0$:

$$\frac{dx}{d\theta} = \frac{a}{2}\left(-8\sin\theta + \operatorname{cosec}^2\theta\right) = 0$$

$$4a\sin\theta = \frac{a}{2\sin^2\theta}$$

$$\sin^3\theta = \frac{1}{8}$$

$$\theta = \pi/6$$

Sub into equation for x :

$$x = \frac{1}{2}a\left(8\cos\frac{\pi}{6} - \cot\frac{\pi}{6} - 1\right)$$

$$= 2a\sqrt{3} - \frac{a\sqrt{3}}{2} - \frac{a}{2}$$

$$x = \frac{3\sqrt{3} - 1}{2}a$$

If this is a maximum, $\frac{d^2x}{d\theta^2} < 0$:

$$\frac{d^2x}{d\theta^2} = \frac{a}{2}\left(-8\cos\frac{\pi}{6} - 2\left(\operatorname{cosec}^2\frac{\pi}{6}\right)\left(\cot\frac{\pi}{6}\right)\right)$$

$$= a\left(-2\sqrt{3} - 4\sqrt{3}\right)$$

$= -6a\sqrt{3} < 0$ as $a > 0$, $\therefore$ maximum

$\therefore$ maximum x value $= \frac{3\sqrt{3} - 1}{2}a$