MODEL ANSWERS

Oxford Cambridge and RSA

Monday 05 October 2020 – Afternoon

AS Level Further Mathematics A

Y531/01 Pure Core

Time allowed: 1 hour 15 minutes

You must have:
- the Printed Answer Booklet
- the Formulae Booklet for AS Level Further Mathematics A
- a scientific or graphical calculator

INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question.
- The acceleration due to gravity is denoted by $g\,\text{m}\,\text{s}^{-2}$. When a numerical value is needed use $g = 9.8$ unless a different value is specified in the question.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION
- The total mark for this paper is **60**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **4** pages.

ADVICE
- Read each question carefully before you start your answer.

1 **In this question you must show detailed reasoning.**

Use an algebraic method to find the square roots of $-77 - 36\text{i}$. **[6]**

$$(a+bi)^2 = \underbrace{a^2 - b^2}_{\text{Real Part} = -77} + \underbrace{2abi}_{\text{Imaginary Part} = -36i}$$

$a^2 - b^2 = -77$

$2ab = -36 \Rightarrow b = \dfrac{-36}{2a} = \dfrac{-18}{a}$

$a^2 - b^2 = a^2 - \left(\dfrac{-18}{a}\right)^2 = -77$

$a^2 - \dfrac{324}{\cancel{a^2}} = -77$

$\times a^2 \quad \cancel{\times a^2} \quad \times a^2$

$a^4 - 324 = -77a^2$

$\therefore a^4 + 77a^2 - 324 = 0$

$(a^2 - 4)(a^2 + 81) = 0$

$a^2 = 4 \qquad a^2 \neq -81$

$\therefore a = \pm\sqrt{4} = \pm 2$

$b^2 = a^2 + 77 = 4 + 77 = 81$

$\therefore b = \mp\sqrt{81} = \mp 9$

$\boxed{\therefore \text{Roots} = 2 - 9i \ \& \ -2 + 9i}$

2 P, Q and T are three transformations in 2-D.

P is a reflection in the x-axis. **A** is the matrix that represents P.

(a) Write down the matrix **A**. **[1]**

Q is a shear in which the y-axis is invariant and the point $\begin{pmatrix}1\\0\end{pmatrix}$ is transformed to the point $\begin{pmatrix}1\\2\end{pmatrix}$. **B** is the matrix that represents Q.

(b) Find the matrix **B**. **[2]**

T is P followed by Q. **C** is the matrix that represents T.

(c) Determine the matrix **C**. **[2]**

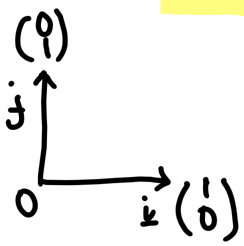
L is the line whose equation is $y = x$.

(d) Explain whether or not L is a line of invariant points under T. **[2]**

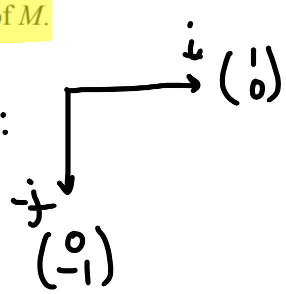
An object parallelogram, M, is transformed under T to an image parallelogram, N.

(e) Explain what the value of the determinant of **C** means about
- the area of N compared to the area of M,
- the orientation of N compared to the orientation of M. **[3]**

(a.)



Reflection in x-axis:



$$\boxed{\therefore \underline{A} = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}}$$

(b.) $\begin{pmatrix}0\\1\end{pmatrix} \rightarrow \begin{pmatrix}0\\1\end{pmatrix}$ means y-axis invariant.

$\begin{pmatrix}1\\0\end{pmatrix} \rightarrow \begin{pmatrix}1\\2\end{pmatrix}$

$$\boxed{\therefore \underline{B} = \begin{pmatrix}1 & 0\\2 & 1\end{pmatrix}}$$

(c.) $T = QP \Rightarrow \underline{C} = \underline{B}\,\underline{A}$

$$\underline{C} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$$

$$\boxed{\therefore \underline{C} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}}$$

(d.) $L: y = x$

$$\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ x \end{pmatrix} = \begin{pmatrix} x \\ 2x - x \end{pmatrix} = \begin{pmatrix} x \\ x \end{pmatrix}$$

Since each point gets mapped to itself, L is a line of invariant points.

(e.) $\det \underline{C} = (1)(-1) - (0)(2) = -1$

$\therefore$ Area of N is same as the area of M.

$\therefore$ Orientation of N is reverse of the orientation of M.

3 In this question you must show detailed reasoning.

The complex number $7 - 4\text{i}$ is denoted by z.

(a) Giving your answers in the form $a + b\text{i}$, where a and b are rational numbers, find the following.

(i) $3z - 4z^*$ **[2]**

(ii) $(z+1-3\text{i})^2$ **[2]**

(iii) $\dfrac{z+1}{z-1}$ **[2]**

(b) Express z in modulus-argument form giving the modulus exactly and the argument correct to 3 significant figures. **[3]**

(c) The complex number ω is such that $z\omega = \sqrt{585}(\cos(0.5) + \text{i}\sin(0.5))$.

Find the following.

- $|\omega|$
- $\arg(\omega)$, giving your answer correct to 3 significant figures **[3]**

(a.) $z = 7 - 4i$

$z^* = 7 + 4i$

(i) $3z - 4z^* = 3(7-4i) - 4(7+4i)$

$= 21 - 12i - 28 - 16i$

$= \boxed{-7 - 28i}$

(ii) $(z+1-3i)^2 = (7 - 4i + 1 - 3i)^2$

$= (8 - 7i)^2$

$= 8^2 - 56i - 56i + 7^2 i^2$

$= 64 - 112i - 49$

$= \boxed{15 - 112i}$

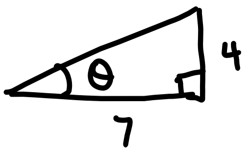
(iii) $\dfrac{z+1}{z-1} = \dfrac{7-4i+1}{7-4i-1} = \dfrac{8-4i}{6-4i} \times \dfrac{6+4i}{6+4i}$

$$= \frac{48+32i-24i-16i^2}{36-16i^2}$$

$$= \frac{64+8i}{52}$$

$$= \boxed{\frac{16}{13} + \frac{2}{13}i}$$

(b.) $|z| = \sqrt{7^2+(-4)^2} = \sqrt{65}$



$\theta = \tan^{-1}\left(-\frac{4}{7}\right) = -0.5191\ldots \approx -0.519$ rads (3sf)

$$\boxed{\therefore z = \sqrt{65}\left(\cos(-0.519) + i\sin(-0.519)\right)}$$

(c.) $zw = \sqrt{585}\left(\cos(0.5) + i\sin(0.5)\right)$

$|w| = \sqrt{585} \div \sqrt{65} = 3$

$\boxed{\therefore |w| = 3}$

$\arg(w) = 0.5 - -0.519 \approx 1.02$ rads (3sf)

$\boxed{\therefore \arg(w) = 1.02}$

4 You are given the system of equations

$$a^2x - 2y = 1$$
$$x + b^2y = 3$$

where a and b are real numbers.

(a) Use a matrix method to find x and y in terms of a and b. **[4]**

(b) Explain why the method used in part **(a)** works for all values of a and b. **[2]**

(a.) $\begin{pmatrix} a^2 & -2 \\ 1 & b^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Let $\underline{A} = \begin{pmatrix} a^2 & -2 \\ 1 & b^2 \end{pmatrix}$

$\det \underline{A} = (a^2)(b^2) - (-2)(1) = a^2b^2 + 2$

$\underline{A}^{-1} = \dfrac{1}{a^2b^2+2}\begin{pmatrix} b^2 & 2 \\ -1 & a^2 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \underline{A}^{-1}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \dfrac{1}{a^2b^2+2}\begin{pmatrix} b^2 & 2 \\ -1 & a^2 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$= \dfrac{1}{a^2b^2+2}\begin{pmatrix} b^2+6 \\ 3a^2-1 \end{pmatrix}$

$$\boxed{\therefore\ x = \frac{b^2+6}{a^2b^2+2},\ y = \frac{3a^2-1}{a^2b^2+2}}$$

(b.) Since $\boxed{a^2b^2 = (ab)^2 \geq 0}$, then $\boxed{a^2b^2+2 > 0}$ for all values of a and b. The determinant of the matrix cannot be 0, so the matrix is never singular.

$\therefore$ Inverse always exists & the method always works.

5 **In this question you must show detailed reasoning.**

$+ \quad - \quad + \quad -$

The cubic equation $5x^3 + 3x^2 - 4x + 7 = 0$ has roots α, β and γ.

Find a cubic equation with integer coefficients whose roots are $\alpha + \beta$, $\beta + \gamma$ and $\gamma + \alpha$. **[7]**

$\rightarrow \alpha + \beta + \gamma = \frac{-3}{5}$

$\rightarrow \alpha\beta + \beta\gamma + \gamma\alpha = \frac{-4}{5}$

$\rightarrow \alpha\beta\gamma = \frac{-7}{5}$

$\rightarrow (\alpha + \beta) + (\beta + \gamma) + (\gamma + \alpha) = 2(\alpha + \beta + \gamma) = 2\left(\frac{-3}{5}\right) = \frac{-6}{5}$

$\rightarrow (\alpha + \beta)(\beta + \gamma) + (\beta + \gamma)(\gamma + \alpha) + (\gamma + \alpha)(\alpha + \beta)$

$= 3(\alpha\beta + \beta\gamma + \gamma\alpha) + \alpha^2 + \beta^2 + \gamma^2$

$= \alpha\beta + \beta\gamma + \gamma\alpha + (\alpha + \beta + \gamma)^2$

$= -\frac{4}{5} + \left(\frac{-3}{5}\right)^2 = -\frac{11}{25}$

$\rightarrow (\alpha + \beta)(\beta + \gamma)(\gamma + \alpha)$

$= (\alpha + \beta)(\beta\gamma + \alpha\beta + \gamma^2 + \alpha\gamma)$

$= \alpha\beta\gamma + \alpha^2\beta + \alpha^2\gamma^2 + \alpha^2\gamma + \beta^2\gamma + \alpha\beta^2 + \beta\gamma^2 + \alpha\beta\gamma$

$= 2\alpha\beta\gamma + \alpha(\alpha\beta + \beta\gamma + \gamma\alpha) - \alpha\beta\gamma + \beta(\alpha\beta + \beta\gamma + \gamma\alpha) - \alpha\beta\gamma$

$+ \gamma(\alpha\beta + \beta\gamma + \gamma\alpha) - \alpha\beta\gamma$

$= (\alpha\beta + \beta\gamma + \gamma\alpha)(\alpha + \beta + \gamma) - \alpha\beta\gamma$

$= \left(\frac{-4}{5}\right)\left(\frac{-3}{5}\right) - \left(\frac{-7}{5}\right) = \frac{47}{25}$

$\boxed{\therefore 25x^3 + 30x^2 - 11x - 47 = 0}$

6 Prove that $n! > 2^{2n}$ for all integers $n \geqslant 9$. **[5]**

① Let $n = 9$:

$LHS = 9! = 362880$

$RHS = 2^{2(9)} = 2^{18} = 262144 < LHS$

$\therefore$ True for $n = 9$.

② Assume true for k, where $k \geq 9$.

$k! > 2^{2k}$

③ Let $n = k+1$:

$(k+1)! = (k+1)k!$

$2^{2(k+1)} = 2^{2k+2} = 2^{2k} \cdot 2^{2}$

$(k+1)k! > (k+1)2^{2k} > 9 \cdot 2^{2k} > 4 \cdot 2^{2k} = 2^{2k+2} = 2^{2(k+1)}$

$\therefore (k+1)! > 2^{2(k+1)}$

④ So true for $n = k \Rightarrow$ true for $n = k+1$.

True for $n = 9$.

$\therefore$ True for all integers $n \geq 9$.

7 The equations of two **intersecting** lines are

$$\mathbf{r} = \begin{pmatrix} -12 \\ a \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \qquad \mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix}$$

where a is a constant.

(a) Find a vector, **b**, which is perpendicular to both lines. **[2]**

(b) Show that $\mathbf{b}.\begin{pmatrix} -12 \\ a \\ -1 \end{pmatrix} = \mathbf{b}.\begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix}$. **[2]**

(c) Hence, or otherwise, find the value of a. **[2]**

(a.) $\underline{b} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix} = \boxed{\begin{pmatrix} -3 \\ -1 \\ 8 \end{pmatrix}}$

since the lines intersect

(b.) $\begin{pmatrix} -12 \\ a \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix}$

$\therefore \begin{pmatrix} -12 \\ a \\ -1 \end{pmatrix} \cdot \underline{b} + \lambda \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \cdot \underline{b} = \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} \cdot \underline{b} + \mu \begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix} \cdot \underline{b}$

$\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \cdot \underline{b} = \begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix} \cdot \underline{b} = 0 \Rightarrow \boxed{\underline{b} \cdot \begin{pmatrix} -12 \\ a \\ -1 \end{pmatrix} = \underline{b} \cdot \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix}}$

(c.) $\begin{pmatrix} -3 \\ -1 \\ 8 \end{pmatrix} \cdot \begin{pmatrix} -12 \\ a \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \\ 8 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix}$

$36 - a - 8 = -6 + 40$

$\therefore a = 36 - 8 + 6 - 40 = -6$

$\boxed{\therefore a = -6}$

8 Two loci, C_1 and C_2, are defined by

$$C_1 = \{z : |z| = |z - 4d^2 - 36|\}$$

$$C_2 = \left\{z : \arg(z - 12d - 3\text{i}) = \tfrac{1}{4}\pi\right\}$$

where d is a real number.

(a) Find, in terms of d, the complex number which is represented on an Argand diagram by the point of intersection of C_1 and C_2.

[You may assume that $C_1 \cap C_2 \neq \varnothing$.] **[6]**

(b) Explain why the solution found in part **(a)** is not valid when $d = 3$. **[2]**

(a.) C_1 is represented by the line $x = 2d^2 + 18$.

C_2 is represented by the half-line $y = x + c$.

$3 = 12d + c$

$y = x + 3 - 12d$

$x = 2d^2 + 18 \Rightarrow y = 2d^2 - 12d + 21$

$\boxed{\therefore 2d^2 + 18 + (2d^2 - 12d + 21)i}$

(b.) When $d = 3$, point of intersection would be $36 + 3i$.

$C_2 = \left\{z : \arg(z - (36 + 3i)) = \frac{1}{4}\pi\right\}$

But $36 + 3i$ is not in C_2 since $\arg 0$ is undefined.