OCR
Oxford Cambridge and RSA

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A Level Further Mathematics A
Y541 Pure Core 2
Sample Question Paper

Date – Morning/Afternoon
Time allowed: 1 hour 30 minutes

OCR supplied materials:
- Printed Answer Booklet
- Formulae A Level Further Mathematics A

You must have:
- Printed Answer Booklet
- Formulae A Level Further Mathematics A
- Scientific or graphical calculator

INSTRUCTIONS

- Use black ink. HB pencil may be used for graphs and diagrams only.
- Complete the boxes provided on the Printed Answer Booklet with your name, centre number and candidate number.
- Answer **all** the questions.
- **Write your answer to each question in the space provided in the Printed Answer Booklet.**
- Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- Do **not** write in the bar codes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The acceleration due to gravity is denoted by $g\,\mathrm{m\,s^{-2}}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION

- The total number of marks for this paper is **75**.
- The marks for each question are shown in brackets **[ ]**.
- **You are reminded of the need for clear presentation in your answers.**
- The Printed Answer Booklet consists of **12** pages. The Question Paper consists of **4** pages.

Answer **all** the questions.

1 Find $\sum_{r=1}^{n}(r+1)(r+5)$. Give your answer in a fully factorised form. **[4]**

$$\sum_{r=1}^{n}(r^2+6r+5)$$

$$=\sum_{r=1}^{n} r^2 + 6\sum_{r=1}^{n} r + \sum_{r=1}^{n} 5$$

As $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$ and $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$:

$$\frac{1}{6}n(n+1)(2n+1) + 6\times\frac{1}{2}n(n+1) + 5n$$

$$=\frac{1}{6}n(n+1)(2n+1) + 3n(n+1) + 5n$$

$$=\frac{1}{6}n\left((n+1)(2n+1) + 18(n+1) + 30\right)$$

$$=\frac{1}{6}n(2n^2+3n+1+18n+18+30) = \frac{1}{6}n(2n^2+21n+49)$$

$$\therefore \sum_{r=1}^{n}(r+1)(r+5) = \frac{1}{6}n(2n+7)(n+7)$$

2 **In this question you must show detailed reasoning.**

The finite region R is enclosed by the curve with equation $y = \dfrac{8}{\sqrt{16+x^2}}$, the x-axis and the lines $x = 0$ and $x = 4$. Region R is rotated through 360° about the x-axis. Find the exact value of the volume generated. **[4]**

$$V = \pi \int_0^4 y^2 \, dx$$

$$= \pi \int_0^4 \left(\frac{8}{\sqrt{16+x^2}} \right)^2 dx$$

$$= \pi \int_0^4 \frac{64}{16+x^2} \, dx = 64\pi \int_0^4 \frac{1}{16+x^2} \, dx$$

Knowing that $\int \frac{1}{a^2+x^2} \, dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right)$

$$\Rightarrow \left[64\pi \times \frac{1}{4} \arctan\left(\frac{x}{4}\right) \right]_0^4 = 16\pi \arctan(1) - 16\pi \arctan(0)$$

$$= 4\pi^2$$

3 (i) Find $\sum_{r=1}^{n}\left(\frac{1}{r}-\frac{1}{r+2}\right)$. **[3]**

(ii) What does the sum in part (i) tend to as $n \to \infty$? Justify your answer. **[1]**

i) $f(1) \quad 1 - \cancel{\frac{1}{3}}$

$f(2) \quad \frac{1}{2} - \cancel{\frac{1}{4}}$

$f(3) \quad \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}$

...

$f(n-1) \quad \cancel{\frac{1}{n-1}} - \frac{1}{n+1}$

$f(n) \quad \cancel{\frac{1}{n}} - \frac{1}{n+2}$

Plugging numbers into $\frac{1}{r} - \frac{1}{r+2}$ and cancelling the terms.

$\Rightarrow 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}$

$= \frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}$ ← this answer would also be valid.

$= \frac{3(n+1)(n+2)}{2(n+1)(n+2)} - \frac{2(n+2)}{2(n+1)(n+2)} - \frac{2(n+1)}{2(n+1)(n+2)}$

$= \frac{3n^2+9n+6-2n-4-2n-2}{2(n+1)(n+2)}$

$= \frac{3n^2+5n}{2(n+1)(n+2)}$

ii) when $n \to \infty$, $\frac{1}{n+1} \to 0$, $\frac{1}{n+2} \to 0$

$\therefore \frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2} \to \frac{3}{2}$

4 It is given that $\dfrac{5x^2+x+12}{x^3+kx} \equiv \dfrac{A}{x} + \dfrac{Bx+C}{x^2+k}$ where k, A, B and C are positive integers.

Determine the set of possible values of k. **[5]**

$$\frac{5x^2+x+12}{x(x^2+K)} \equiv \frac{A}{x} + \frac{Bx+C}{x^2+K}$$

$$\Rightarrow 5x^2+x+12 = A(x^2+K) + (Bx+C)x$$

$$5x^2+x+12 = Ax^2+AK+Bx^2+Cx$$

compare the coefficient:

x^2 coeff: $5 = A+B$

x coeff: $1 = C$

constant: $12 = AK \quad \Rightarrow \frac{12}{K} = A$

$$5 = \frac{12}{K} + B$$

$$B = 5 - \frac{12}{K}$$

Since A and B must be integers, K must a factor of 12.

$B > 0$, gives $0 < 5 - \frac{12}{K}$

$$\frac{12}{K} < 5$$

$\therefore$ $K = 3, 4, 6$ or 12

5 **In this question you must show detailed reasoning.**

Evaluate $\int_0^{\infty} 2xe^{-x}\,dx$.

[You may use the result $\lim_{x\to\infty} xe^{-x} = 0$.] **[4]**

$$\lim_{t\to\infty} \int_0^t 2xe^{-x}\,dx$$

Using $\int f(x)\,dx = uv - \int u'v\,dx$

$u = x$ $\quad v' = 2e^{-x}$

$u' = 1$ $\quad v = -2e^{-x}$

$$= \lim_{t\to\infty} \left[-2xe^{-x}\right]_0^t + \int_0^t 2e^{-x}\,dx$$

$$= \lim_{t\to\infty} \left[-2xe^{-x} - 2e^{-x}\right]_0^t$$

$$= (-2te^{-t} - 2e^{-t}) - (0 - 2)$$

$$= -2te^{-t} - 2e^{-t} + 2$$

As $t \to \infty$, $-2te^{-t} \to 0$, $-2e^{-t} \to 0$

$\therefore \int_0^{\infty} 2xe^{-x}\,dx \to 2$

6 The equation of a plane Π is $x-2y-z=30$.

(i) Find the acute angle between the line $\mathbf{r}=\begin{pmatrix}3\\2\\-5\end{pmatrix}+\lambda\begin{pmatrix}-5\\3\\2\end{pmatrix}$ and Π. **[4]**

(ii) Determine the geometrical relationship between the line $\mathbf{r}=\begin{pmatrix}1\\4\\2\end{pmatrix}+\mu\begin{pmatrix}3\\-1\\5\end{pmatrix}$ and Π. **[4]**

Acute angle between a line and a plane means we use $\sin\theta=\left|\frac{a.n}{|a||n|}\right|$

a is the direction vector $=\begin{pmatrix}-5\\3\\2\end{pmatrix}$

n is the normal vector to the plane $=\begin{pmatrix}1\\-2\\-1\end{pmatrix}$

$$a.n=\begin{pmatrix}-5\\3\\2\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\-1\end{pmatrix}=-5-6-2=-13$$

$$|a|=\sqrt{(-5)^2+3^2+2^2}=\sqrt{38}$$

$$|n|=\sqrt{1^2+(-2)^2+(-1)^2}=\sqrt{6}$$

$$\theta=\sin\left(\left|\frac{-13}{\sqrt{38}\times\sqrt{6}}\right|\right)=59.4^\circ\ (3sf)$$

ii) $\begin{pmatrix}3\\-1\\5\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\-1\end{pmatrix}=3+2-5=0$

The line is in or parallel to the plane.

Now check if the line is <u>in</u> the plane.

$x-2y-z=30$

$(1)-2(4)-(2)=-9$

$-9\neq30$

Point on the line is not in the plane

So line is parallel to the plane.

7 **(i)** Use the Maclaurin series for $\sin x$ to work out the series expansion of $\sin x \sin 2x \sin 4x$ up to and including the term in x^3. **[4]**

(ii) Hence find, in exact surd form, an approximation to the least positive root of the equation $2\sin x \sin 2x \sin 4x = x$. **[3]**

i)

Since $\sin x = x - \dfrac{x^3}{3!} + \ldots$

$$\sin 2x = 2x - \frac{(2x)^3}{3!} + \ldots$$

$$\sin 4x = 4x - \frac{(4x)^3}{3!} + \ldots$$

$$\therefore \sin x \sin 2x \sin 4x = \left(x - \frac{x^3}{3!}\right)\left(2x - \frac{8x^3}{3!}\right)\left(4x - \frac{64x^3}{3!}\right)$$

$$= \left(x - \frac{x^3}{6}\right)\left(2x - \frac{4x^3}{3}\right)\left(4x - \frac{32x^3}{3}\right)$$

$$= 8x^3 - 28x^5$$

ii) $2 \times (8x^3 - 28x^5) = x$

$$56x^5 - 16x^3 + x = 0$$

$$x(56x^4 + 16x^2 + 1) = 0$$

$$x^2 = \frac{16 \pm \sqrt{32}}{112}$$

$$x = \sqrt{\frac{4 - \sqrt{2}}{28}}$$ ← least positive root.

8 The equation of a curve is $y = \cosh^2 x - 3\sinh x$. Show that $\left(\ln\left(\frac{3+\sqrt{13}}{2}\right), -\frac{5}{4}\right)$ is the only stationary point on the curve. **[8]**

First differentiation $y = \cosh^2 x - 3\sinh x$

$$\frac{dy}{dx} = 2\cosh x \sinh x - 3\cosh x$$

At a stationary point, $\frac{dy}{dx} = 0$

$$2\cosh x \sinh x - 3\cosh x = 0$$

$$\cosh x (2\sinh x - 3) = 0$$

$\cosh x = 0$ has no roots, as $\cosh x \geqslant 1$

sinh is a one to one function so $\therefore$ just one stationary point when $2\sinh x - 3 = 0$

$$\sinh x = \frac{3}{2}$$

$$x = \operatorname{arsinh} \frac{3}{2}$$

$$= \ln\left(\frac{3}{2} + \sqrt{1 + \left(\frac{3}{2}\right)^2}\right)$$

$$= \ln\left(\frac{3+\sqrt{13}}{2}\right)$$

$$\cosh x = \sqrt{1+\sinh^2 x} = \sqrt{1+\frac{9}{4}} = \sqrt{\frac{13}{4}}$$

$$y = \cosh^2 x - 3\sinh x$$

$$= \frac{13}{4} - 3 \times \frac{3}{2}$$

$$= -\frac{5}{4}$$

$\therefore$ coordinates $\left(\ln\left(\frac{3+\sqrt{13}}{2}\right), -\frac{5}{4}\right)$ (as required)

9 A curve has equation $x^4 + y^4 = x^2 + y^2$, where x and y are not both zero.

(i) Show that the equation of the curve in polar coordinates is $r^2 = \dfrac{2}{2 - \sin^2 2\theta}$. **[4]**

(ii) Deduce that no point on the curve $x^4 + y^4 = x^2 + y^2$ is further than $\sqrt{2}$ from the origin. **[2]**

i) Remember:
$x = r\cos\theta$
$y = r\sin\theta$
$x^2 + y^2 = r^2$

$$(r\cos\theta)^4 + (r\sin\theta)^4 = r^2$$
$$r^4(\cos^4\theta + \sin^4\theta) = r^2$$
$$r^2(\cos^4\theta + \sin^4\theta) = 1$$
$$r^2 = \frac{1}{\cos^4\theta + \sin^4\theta}$$

$$r^2 = \frac{1}{(\cos^2\theta + \sin^2\theta) - 2\sin^2\theta\cos^2\theta}$$
$$= \frac{1}{1 - \frac{1}{2}\sin^2 2\theta}$$

$\leftarrow \sin^2 2\theta = 4\cos^2\theta\sin^2\theta$

$$= \boxed{\frac{2}{2 - \sin^2\theta}}$$ (as required)

ii) Maximum value of r occurs when $\sin 2\theta = 1$

$$r^2 = \frac{2}{2-1} = 2$$

$\Rightarrow \boxed{r = \sqrt{2}}$

10 Let $C = \sum_{r=0}^{20} \binom{20}{r} \cos r\theta$. Show that $C = 2^{20} \cos^{20}\left(\frac{1}{2}\theta\right)\cos 10\theta$. **[8]**

Let $S = \sum_{r=1}^{20} \binom{20}{r} \sin r\theta$

$C + Si = 1 + 20(\cos\theta + i\sin\theta) + \binom{20}{2}(\cos 2\theta + i\sin 2\theta) + \ldots + (\cos 20\theta + i\sin 20\theta)$

By de Moivre's Theorem

$C + Si = 1 + 20z + \binom{20}{2}z^2 + \ldots + z^{20}$

where $z = \cos\theta + i\sin\theta$

$\Rightarrow \quad (1+z)^{20}$

$\cos\theta = 2\cos^2\frac{\theta}{2} - 1$

$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$\therefore \quad (1+z)^{20} = \left(1 + 2\cos^2\frac{\theta}{2} - 1 + i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^{20}$

$= \left(2\cos^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^{20}$

$= \left(2\cos\frac{\theta}{2}\right)^{20}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)^{20}$

$= \left(2\cos^{20}\frac{\theta}{2}\right)\left(\cos 10\theta + i\sin 10\theta\right)$

Re: $C = 2^{20}\cos^{20}\frac{\theta}{2}\cos 10\theta$ (as required)

11 During an industrial process substance X is converted into substance Z. Some of the substance X goes through an intermediate phase, and is converted to substance Y, before being converted to substance Z. The situation is modelled by

$$\frac{dy}{dt} = 0.3x - 0.2y \text{ and } \frac{dz}{dt} = 0.2y + 0.1x$$

where x, y and z are the amounts in kg of X, Y and Z at time t hours after the process starts.

Initially there is 10 kg of substance X and nothing of substances Y and Z. The amount of substance X decreases exponentially. The initial rate of decrease is 4 kg per hour.

(i) Show that $x = Ae^{-0.4t}$, stating the value of A. **[3]**

(ii) (a) Show that $\frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt} = 0$. **[2]**

(b) Comment on this result in the context of the industrial process. **[2]**

(iii) Express y in terms of t. **[5]**

(iv) Determine the maximum amount of substance Y present during the process. **[3]**

(v) How long does it take to produce 9 kg of substance Z? **[2]**

i) $A = 10$ (starting mass of X)

$x = Ae^{kt}$

$\frac{dx}{dt} = kAe^{kt}$

As initial rate of decrease is 4 kg/hr

$-4 = Ak$

$-4 = 10k$

$-0.4 = k$ (as required)

iia) $\frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt} = -4e^{-0.4t} + (0.3x - 0.2y) + (0.2y + 0.1x)$

$= -0.4x + 0.3x + 0.1x = 0$

$\therefore \frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt} = 0$ (as required) iib) $\frac{d}{dt}(x+y+z)=0$

$x+y+z$ must be constant.
Throughout the industrial process the total amount of the three substances is constant.
(no loss or gains of amounts)

iii) Substitute x and rearrange:

$$\frac{dy}{dt}+0.2y=3e^{-0.4t}$$

I.F. $e^{\int 0.2\,dt}=e^{0.2t}$

$$\frac{d}{dt}\left(e^{0.2t}y\right)=3e^{-0.2t}$$

$$e^{0.2t}y=3\int e^{-0.2t}\,dt$$

$$e^{0.2}y=-15e^{-0.2t}+c$$

$$y=-15e^{-0.4t}+ce^{-0.2}$$

when $y=0$, $t=0$ $\quad \underline{c=15}$

$$\therefore\ y=-15e^{-0.4t}+15e^{-0.2t}$$

iv) $\frac{dy}{dt}=0$

$$0.3x-0.2y=0$$

$$0.3x=0.2y$$

$$0.3\left(10e^{-0.4t}\right)=0.2\left(-15e^{-0.4t}+15e^{-0.2t}\right)$$

$$3e^{-0.4t}=-3e^{-0.4t}+3e^{-0.2t}$$

$$3e^{-0.2t}=6e^{-0.4t}$$

$$e^{-0.2t}=2e^{-0.4t}$$

$$e^{0.2t}=2$$

$$0.2t=\ln 2$$

$t=5\ln 2=3.466$ (3dp)

$$\therefore\ y_{max}=-15e^{-0.4(5\ln 2)}+15e^{-0.2(5\ln 2)}$$

$$=3.75\,kg$$

v) 9kg of substance Z implies $x+y=1$

$$10e^{-0.4t} - 15e^{-0.4t} + 15e^{-0.2t} = 1$$

$$5e^{-0.4t} - 15e^{-0.2t} + 1 = 0 \qquad \times e^{0.4t}$$

$$5 - 15e^{0.2t} + e^{0.4t} = 0$$

$$e^{0.4t} - 15e^{0.2t} + 5 = 0$$

Using quadratic formula:

$$e^{0.2t} = 14.658\ldots$$

$t = 13.4$ hours (3sf)