OCR
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MODEL ANSWERS

Thursday 08 October 2020 – Afternoon

A Level Further Mathematics A Y541/01 Pure Core 2

Time allowed: 1 hour 30 minutes

You must have:
- the Printed Answer Booklet
- the Formulae Booklet for A Level Further Mathematics A
- a scientific or graphical calculator

INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question.
- The acceleration due to gravity is denoted by $g\,\text{m}\,\text{s}^{-2}$. When a numerical value is needed use $g = 9.8$ unless a different value is specified in the question.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION
- The total mark for this paper is **75**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **8** pages.

ADVICE
- Read each question carefully before you start your answer.

1 **In this question you must show detailed reasoning.**

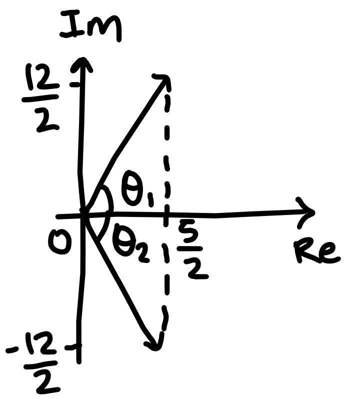
Solve the equation $4z^2 - 20z + 169 = 0$. Give your answers in modulus-argument form. **[5]**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 4$, $b = -20$, $c = 169$

$$z = \frac{--20 \pm \sqrt{(-20)^2 - 4(4)(169)}}{2(4)} = \frac{5 \pm 12i}{2}$$

$$|z| = \sqrt{\left(\frac{5}{2}\right)^2 + \left(\frac{12}{2}\right)^2} = \frac{13}{2}$$



$\theta_1 = \tan^{-1}\left(\frac{12 \div 2}{5 \div 2}\right) = 1.1760\ldots \approx 1.18$ rad (3sf)

$\theta_2 = -1.18$ rad (3sf)

$$\boxed{\begin{aligned} &\therefore z = \frac{13}{2}(\cos 1.18 + i\sin 1.18) \\ &\text{OR } z = \frac{13}{2}(\cos(-1.18) + i\sin(-1.18)) \end{aligned}}$$

2 **In this question you must show detailed reasoning.**

$+ \quad - \quad + \quad -$

The roots of the equation $3x^3 - 2x^2 - 5x - 4 = 0$ are α, β and γ.

(a) Find a cubic equation with integer coefficients whose roots are α^2, β^2 and γ^2. **[4]**

(b) Find the exact value of $\dfrac{\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2}{\alpha\beta\gamma}$. **[2]**

(a.) $\alpha^2\beta^2\gamma^2 = (\alpha\beta\gamma)^2 = \left(-\frac{-4}{3}\right)^2 = \frac{16}{9}$

$$\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2(\alpha\beta\gamma)(\alpha+\beta+\gamma)$$
$$= \left(\frac{-5}{3}\right)^2 - 2\left(-\frac{-4}{3}\right)\left(-\frac{-2}{3}\right)$$
$$= \frac{44}{9}$$

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$
$$= \left(-\frac{-2}{3}\right)^2 - 2\left(\frac{-5}{3}\right)$$
$$= \frac{34}{9}$$

$u^3 - \frac{34}{9}u^2 + u - \frac{16}{9} = 0 \Rightarrow \boxed{\therefore 9u^3 - 34u^2 + 9u - 16 = 0}$

(b.) $\dfrac{\sum \alpha^2\beta^2}{\alpha\beta\gamma} = \dfrac{1}{\left(-\frac{-4}{3}\right)} = \boxed{\frac{3}{4}}$

3 **In this question you must show detailed reasoning.**

(a) Use partial fractions to show that $\sum_{r=5}^{n} \frac{3}{r^2+r-2} = \frac{37}{60} - \frac{1}{n} - \frac{1}{n+1} - \frac{1}{n+2}$. **[5]**

(b) Write down the value of $\lim_{n\to\infty}\left(\sum_{r=5}^{n} \frac{3}{r^2+r-2}\right)$. **[1]**

(a.) $\frac{3}{r^2+r-2} \equiv \frac{A}{r-1} + \frac{B}{r+2}$

$A(r+2) + B(r-1) \equiv 3 \Rightarrow$ ① $A + \cancel{B} = 0$

$+$ ② $2A - \cancel{B} = 3$

$3A = 3 \Rightarrow \therefore A = 1$

$\therefore B = -A = -1$

$\therefore \sum_{r=5}^{n} \frac{3}{r^2+r-2} = \sum_{r=5}^{n} \frac{1}{r-1} - \frac{1}{r+2}$

$r=5: \frac{1}{5-1} - \frac{1}{5+2} = \frac{1}{4} - \cancel{\frac{1}{7}}$

$r=6: \frac{1}{6-1} - \frac{1}{6+2} = \frac{1}{5} - \cancel{\frac{1}{8}}$

$r=7: \frac{1}{7-1} - \frac{1}{7+2} = \frac{1}{6} - \cancel{\frac{1}{9}}$

$\dots$

$r=n-2: \frac{1}{n-2-1} - \frac{1}{n-2+2} = \cancel{\frac{1}{n-3}} - \frac{1}{n}$

$r=n-1: \frac{1}{n-1-1} - \frac{1}{n-1+2} = \cancel{\frac{1}{n-2}} - \frac{1}{n+1}$

$r=n: \cancel{\frac{1}{n-1}} - \frac{1}{n+2}$

$$\therefore \sum_{r=5}^{n} \frac{3}{r^2+r-2} = \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{n} - \frac{1}{n+1} - \frac{1}{n+2}$$

$$= \boxed{\frac{37}{60} - \frac{1}{n} - \frac{1}{n+1} - \frac{1}{n+2}}$$

(b.) $$\lim_{n \to \infty} \left(\sum_{r=5}^{n} \frac{3}{r^2+r-2} \right) = \boxed{\frac{37}{60}}$$

4 The equations of two intersecting lines l_1 and l_2 are

$$l_1: \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ a \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \qquad l_2: \mathbf{r} = \begin{pmatrix} 7 \\ 9 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

where a is a constant.

The equation of the plane Π is

$$\mathbf{r}.\begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix} = -14.$$

l_1 and Π intersect at Q.

l_2 and Π intersect at R.

(a) Verify that the coordinates of R are (13, 3, –14). **[2]**

(b) Determine the exact value of the length of QR. **[7]**

(a.) $\begin{pmatrix} 13 \\ 3 \\ -14 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix} = 13 + 15 - 42 = -14$

$\boxed{\therefore R \text{ is on } \pi \quad \therefore R = (13, 3, -14)}$

(b.) Since lines intersect:

$$\begin{pmatrix} 1 \\ 0 \\ a \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 7 \\ 9 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

$1 + 2\lambda = 7 - \mu$

$\lambda = 9 + \mu$

$a - 3\lambda = -2 + 2\mu$

$\hookrightarrow \therefore a = 3\lambda + 2\mu - 2 = 5$

$\Rightarrow$ ① $2\lambda + \cancel{\mu} = 6$

$+$ ② $\lambda - \cancel{\mu} = 9$

$3\lambda = 15 \Rightarrow \therefore \lambda = 5$

$\therefore \mu = 5 - 9 = -4$

At Q : $\begin{pmatrix} 1+2\lambda \\ 0+\lambda \\ 5-3\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix} = -14$

$$1+2\lambda+5\lambda+3(5-3\lambda) = -14$$

$$1+2\lambda+5\lambda+15-9\lambda = -14$$

$$-2\lambda = -30$$

$$\therefore \lambda = \frac{-30}{-2} = 15$$

$\lambda = 15 \Rightarrow Q(31, 15, -40)$

Since $R(13, 3, -14)$:

$$QR = \sqrt{(31-13)^2+(15-3)^2+(-40--14)^2}$$

$$= \sqrt{18^2+12^2+26^2}$$

$$= \sqrt{1144}$$

$$\boxed{\therefore QR = \sqrt{1144}}$$

5 A capacitor is an electrical component which stores charge. The value of the charge stored by the capacitor, in suitable units, is denoted by Q. The capacitor is placed in an electrical circuit.

At any time t seconds, where $t \geqslant 0$, Q can be modelled by the differential equation

$$\frac{d^2Q}{dt^2} - 2\frac{dQ}{dt} - 15Q = 0.$$

Initially the charge is 100 units and it is given that Q tends to a finite limit as t tends to infinity.

(a) Determine the charge on the capacitor when $t = 0.5$. **[6]**

(b) Determine the finite limit of Q as t tends to infinity. **[1]**

(a.) $m^2 - 2m - 15 = 0 \Rightarrow m = 5, -3$

General Solution : $Q = Ae^{-3t} + Be^{5t}$

As $t \to \infty \Rightarrow B = 0$

$t = 0, Q = 100 \Rightarrow A = 100$

$\therefore Q = 100e^{-3t}$

$t = 0.5 \Rightarrow Q = 100e^{-3(0.5)} = 22.31\ldots \approx 22.3$ (3sf)

$\boxed{\therefore Q = 22.3}$

(b.) As $t \to \infty$, $e^{-3t} \to 0$ so Q tends to 0.

6 The equation of a curve in polar coordinates is $r = \ln(1 + \sin\theta)$ for $\alpha \leqslant \theta \leqslant \beta$ where α and β are non-negative angles. The curve consists of a single closed loop through the pole.

(a) By solving the equation $r = 0$, determine the smallest possible values of α and β. [2]

(b) Find the area enclosed by the curve, giving your answer to 4 significant figures. [2]

(c) Hence, by considering the value of r at $\theta = \dfrac{\alpha+\beta}{2}$, show that the loop is **not** circular. [2]

(a.) $r = 0 \Rightarrow \cancel{\ln}(1+\sin\theta) = 0$

$\cancel{e^{\square}} \qquad e^{\square}$

$\cancel{1} + \sin\theta = \cancel{1}$

$\therefore \sin\theta = 0$

$\theta = \sin^{-1}(0) = 0$

$\pi - 0 = 0$

$\boxed{\therefore \alpha = 0,\ \beta = \pi}$

(b.) $A = \frac{1}{2}\int r^2 \, d\theta$

$r^2 = [\ln(1+\sin\theta)]^2$

$A = \frac{1}{2}\int_0^{\pi} (\ln(1+\sin\theta))^2 \, d\theta = 0.41621\ldots \approx 0.4162$ (4sf)

$\boxed{\therefore A = 0.4162}$

(c.) $\theta = \frac{\alpha+\beta}{2} = \frac{0+\pi}{2} = \frac{\pi}{2} \Rightarrow r = \ln\left(1+\sin\frac{\pi}{2}\right) = 0.6931$ (4sf)

$\boxed{\therefore \text{Diameter of Circle, } D = 0.6931}$

But $\underline{A = 0.4162} \Rightarrow 0.4162 = \pi\left(\frac{D}{2}\right)^2 \Rightarrow \underline{D = 0.7280}$ (4sf)

$0.6931 \neq 0.7280$ <u>$\therefore$ Curve is not circular.</u>

7 The matrix $\mathbf{A}$ is given by $\mathbf{A} = \begin{pmatrix} 0.6 & 2.4 \\ -0.8 & 1.8 \end{pmatrix}$.

(a) Find $\det \mathbf{A}$. **[1]**

The matrix $\mathbf{A}$ represents a stretch parallel to one of the coordinate axes followed by a rotation about the origin.

(b) By considering the determinants of these transformations, determine the scale factor of the stretch. **[2]**

(c) Explain whether the stretch is parallel to the x-axis or the y-axis, justifying your answer. **[1]**

(d) Find the angle of rotation. **[2]**

(a.) $\det \underline{A} = (0.6)(1.8) - (2.4)(-0.8) = 3$

$\boxed{\therefore \det \underline{A} = 3}$

(b.) Determinant of Rotation = 1

Determinant of Rotation × Determinant of Stretch = 3

↳ $1 \times sf = 3$

$\boxed{\therefore sf = 3}$

(c.) Since the 2nd column of $\underline{A}$ contains entries bigger than 1 in magnitude, the stretch must be parallel to the y-axis.

(d.) $\underline{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$\cos\theta = 0.6$

$\theta = \cos^{-1}(0.6) = 53.13\ldots \approx 53.1^\circ$ (3sf)

$\boxed{\therefore \text{Angle of Rotation} = 53.1^\circ \text{ clockwise}}$

8 In this question you must show detailed reasoning.

The complex number $-4+\mathrm{i}\sqrt{48}$ is denoted by z.

(a) Determine the cube roots of z, giving the roots in exponential form. **[6]**

The points which represent the cube roots of z are denoted by A, B and C and these form a triangle in an Argand diagram.

(b) Write down the angles that any lines of symmetry of triangle ABC make with the positive real axis, justifying your answer. **[3]**

(a.) $z = -4 + i\sqrt{48}$

$|z| = \sqrt{(-4)^2 + (\sqrt{48})^2} = 8$

$r\cos\theta = -4$

$r\sin\theta = \sqrt{48}$

$\Rightarrow \dfrac{\cancel{r}\sin\theta}{\cancel{r}\cos\theta} = \dfrac{\sqrt{48}}{-4} \Rightarrow \tan\theta = -\sqrt{3}$

$\tan^{-1}(-\sqrt{3}) = -\dfrac{\pi}{3}$

$\sqrt[3]{8} = 2$

$\therefore \theta = -\dfrac{\pi}{3} + \pi = \dfrac{2}{3}\pi$

$\dfrac{2\pi}{3} + 2\pi k$ for $k=1$ is $\dfrac{8}{3}\pi$ & for $k=2$ is $\dfrac{14}{3}\pi$.

$\dfrac{2}{3}\pi \div 3 = \dfrac{2}{9}\pi$

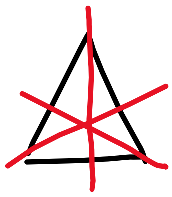
$\dfrac{8}{3}\pi \div 3 = \dfrac{8}{9}\pi$

$\dfrac{14}{3}\pi \div 3 = \dfrac{14}{9}\pi$

$\boxed{\therefore \text{Roots: } 2^{\frac{2}{9}\pi i},\ 2^{\frac{8}{9}\pi i},\ 2^{\frac{14}{9}\pi i}}$

(b.) The cube roots form an equilateral triangle.

$\therefore$ There are 3 lines of symmetry.

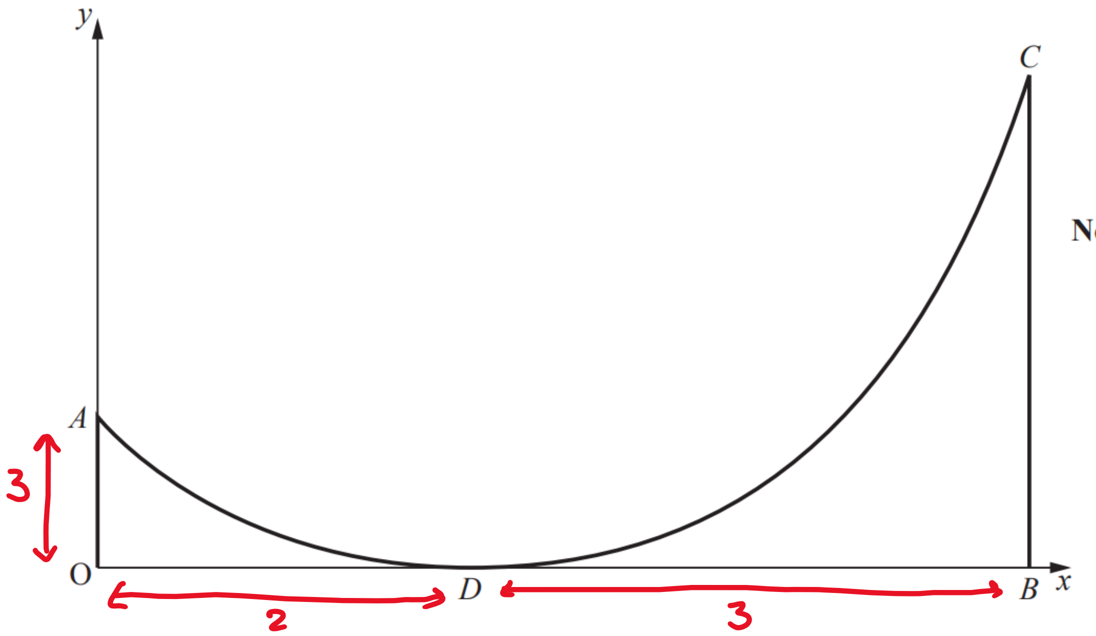


$\boxed{\therefore \theta = \dfrac{2\pi}{9},\ \dfrac{8\pi}{9},\ \dfrac{14\pi}{9}}$

9 Two thin poles, OA and BC, are fixed vertically on horizontal ground. A chain is fixed at A and C such that it touches the ground at point D as shown in the diagram.

On a coordinate system the coordinates of A, B and D are $(0, 3)$, $(5, 0)$ and $(2, 0)$.

A B D



It is required to find the height of pole BC by modelling the shape of the curve that the chain forms.

Jofra models the curve using the equation $y = k\cosh(ax - b) - 1$ where k, a and b are positive constants.

(a) Determine the value of k. **[2]**

(b) Find the exact value of a and the exact value of b, giving your answers in logarithmic form. **[5]**

Holly models the curve using the equation $y = \frac{3}{4}x^2 - 3x + 3$.

(c) Write down the coordinates of the point, (u, v) where u and v are both non-zero, at which the two models will agree. **[1]**

(d) Show that Jofra's model and Holly's model disagree in their predictions of the height of pole BC by 3.32 m to 3 significant figures. **[3]**

(a.) Min value of cosh = 1

Point on ground is at the minimum.

$0 = k(1) - 1 \Rightarrow \boxed{\therefore k = 1}$

(b.) $A(0,3) \Rightarrow 3 = \cosh(a(0) - b) - 1$

$4 = \cosh(-b)$

$-b = \text{arcosh}(4)$ $\quad$ $\text{arcosh}\, x = \ln\left|x + \sqrt{x^2-1}\right|$

$\therefore b = -\text{arcosh}(4) = \ln|4 + \sqrt{15}|$

$D(2,0) \Rightarrow 0 = \cosh(a(2) - b) - 1$

$1 = \cosh(2a - b)$

$2a - b = \text{arcosh}(1) = 0$

$\therefore a = \frac{1}{2}b = \frac{1}{2}\ln|4+\sqrt{15}|$

$$\boxed{\therefore a = \frac{1}{2}\ln|4+\sqrt{15}|,\ b = \ln|4+\sqrt{15}|}$$

(c.) $J: y = \cosh\left(-\frac{1}{2}x\ln|4+\sqrt{15}| + \ln|4+\sqrt{15}|\right) - 1$

$H: y = \frac{3}{4}x^2 - 3x + 3$

By symmetry of both $(u,v) = \boxed{(4,3)}$

(d.) $x = 5 \Rightarrow d_J = \cosh(5a - b) - 1 = 10.0679\ldots$

$d_H = \frac{3}{4}(5)^2 - 3(5) + 3 = 6.75$

$d_J - d_H = 10.0679\ldots - 6.75 = 3.317\ldots \approx 3.32$ (3sf)

$$\boxed{\therefore \text{Difference} = 3.32}$$

10 Let $f(x) = \sin^{-1}(x)$.

(a) (i) Determine $f''(x)$. **[2]**

(ii) Determine the first two non-zero terms of the Maclaurin expansion for $f(x)$. **[3]**

(iii) By considering the first two non-zero terms of the Maclaurin expansion for $f(x)$, find an approximation to $\int_0^{\frac{1}{2}} f(x)\,dx$. Give your answer correct to 6 decimal places. **[2]**

(b) By writing $f(x)$ as $\sin^{-1}(x) \times 1$, determine the value of $\int_0^{\frac{1}{2}} f(x)\,dx$. Give your answer in exact form. **[3]**

(a.) (i) $f(x) = \arcsin(x)$

$$f'(x) = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}}$$

$$f''(x) = -\frac{1}{2}(-2x)(1-x^2)^{-\frac{3}{2}} = x(1-x^2)^{-\frac{3}{2}}$$

$$\boxed{\therefore f''(x) = \frac{x}{(1+x^2)^{3/2}}}$$

(ii) $f(0) = 0$

$f'(0) = 1$

$f''(0) = 0$

$$f'''(x) = \frac{(1-x^2)^{\frac{3}{2}} - 3x^2(1-x^2)^{\frac{1}{2}}}{(1-x^2)^3}$$

$u = x \qquad v = (1-x^2)^{\frac{3}{2}}$

$u' = 1 \qquad v' = \frac{3}{2}(-2x)(1-x^2)^{\frac{1}{2}} = 3x(1-x^2)^{\frac{1}{2}}$

$v^2 = (1-x^2)^3$

$f'''(0) = 1$

$$f(x) = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(1) = x + \frac{x^3}{6} + \ldots$$

$$\boxed{\therefore f(x) = x + \frac{x^3}{6} + \ldots}$$

(iii) $\int_0^{\frac{1}{2}} f(x)\,dx = \int_0^{\frac{1}{2}} x + \frac{x^3}{6}\,dx$

$= \left[\frac{x^2}{2} + \frac{x^4}{24}\right]_0^{\frac{1}{2}}$

$= \left[\frac{0.5^2}{2} - \frac{0.5^4}{24}\right] - 0$

$= \frac{49}{384}$

≈ 0.127604 (6dp)

$$\boxed{\therefore \int_0^{\frac{1}{2}} f(x) = 0.127604}$$

(b.) $\int_0^{\frac{1}{2}} \sin^{-1}x \times 1\,dx = \left[x\sin^{-1}x\right]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}\,dx$

$u = \sin^{-1}x$ — $v = x$

$u' = \frac{1}{\sqrt{1-x^2}}$ $v' = 1$

$y' = x(1-x^2)^{-\frac{1}{2}}$

Let $u = (1-x^2)^{\frac{1}{2}}$

$u' = \frac{1}{2}(-2x)(1-x^2)^{-\frac{1}{2}}$

$= -x(1-x^2)^{-1/2}$

$\therefore y = -(1-x^2)^{1/2}$

$= \left[x\sin^{-1}x + (1-x^2)^{\frac{1}{2}}\right]_0^{\frac{1}{2}}$

$= \left[\frac{\pi}{12} + \frac{\sqrt{3}}{2}\right] - [0-1]$

$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} + 1$

$$\boxed{\therefore \int_0^{\frac{1}{2}} f(x) = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1}$$