OCR
Oxford Cambridge and RSA

Thursday 6 June 2019 – Afternoon

A Level Further Mathematics A

Y541/01 Pure Core 2

Time allowed: 1 hour 30 minutes

You must have:
- Printed Answer Booklet
- Formulae A Level Further Mathematics A

You may use:
- a scientific or graphical calculator

Model Solutions

INSTRUCTIONS
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Answer **all** the questions.
- **Write your answer to each question in the space provided in the Printed Answer Booklet.** If additional space is required, you should use the lined page(s) at the end of the Printed Answer Booklet. The question number(s) must be clearly shown.
- You are permitted to use a scientific or graphical calculator in this paper.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question.
- The acceleration due to gravity is denoted by $g\,\text{m}\,\text{s}^{-2}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION
- The total mark for this paper is **75**.
- The marks for each question are shown in brackets **[ ]**.
- **You are reminded of the need for clear presentation in your answers.**
- The Printed Answer Booklet consists of **16** pages. The Question Paper consists of **8** pages.

Answer **all** the questions.

1 **In this question you must show detailed reasoning.**

(a) By using partial fractions show that $\sum_{r=1}^{n} \frac{1}{r^2+3r+2} = \frac{1}{2} - \frac{1}{n+2}$. **[5]**

(b) Hence determine the value of $\sum_{r=1}^{\infty} \frac{1}{r^2+3r+2}$. **[2]**

a) $\frac{1}{(r+2)(r+1)} = \frac{A}{r+2} + \frac{B}{r+1}$

$1 = A(r+1) + B(r+2)$

when $r=-1$:

$1 = B$

when $r=-2$:

$1 = -A$

$A = -1$

$\Rightarrow \frac{1}{r+1} - \frac{1}{r+2}$

when:

$r=1$ $\quad \frac{1}{2} - \cancel{\frac{1}{3}}$

$r=2$ $\quad \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}$

$r=3$ $\quad \cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}$

$r=n-1$ $\quad \cancel{\frac{1}{n}} - \cancel{\frac{1}{n+1}}$

$r=n$ $\quad \cancel{\frac{1}{n+1}} - \frac{1}{n+2}$

$\therefore \sum_{r=1}^{n} \frac{1}{r^2+3r+2} = \frac{1}{2} - \frac{1}{n+2}$

b) $\sum_{r=1}^{\infty} \frac{1}{r^2+3r+2} = \frac{1}{2}$

As $n \to \infty$, $\frac{1}{n+2} \to 0$

2 **(a)** A plane Π has the equation $\mathbf{r}.\begin{pmatrix} 3 \\ 6 \\ -2 \end{pmatrix} = 15$. C is the point $(4, -5, 1)$.

Find the shortest distance between Π and C. **[3]**

(b) Lines l_1 and l_2 have the following equations.

$$l_1: \mathbf{r} = \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 4 \\ -2 \end{pmatrix}$$

$$l_2: \mathbf{r} = \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Find, in exact form, the distance between l_1 and l_2. **[5]**

a) Using equation

$$\frac{|n_1\alpha + n_2\beta + n_3\gamma + d|}{\sqrt{n_1^2+n_2^2+n_3^2}}$$

$$= \frac{|(4\times3)+(6\times-5)+(-2\times1)-15|}{\sqrt{3^2+6^2+(-2)^2}} = \frac{35}{7} = 5$$

b) $\begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$

$$\frac{\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}{\sqrt{1^2+(-1)^2+3^2} \times \sqrt{1^2+(-2)^2+1^2}} = \cos\theta$$

$\Leftarrow$ using formula $\cos\theta = \frac{a.b}{|a|\cdot|b|}$

$$\cos\theta = \pm \frac{6}{\sqrt{66}}$$

As $d = |r|\sin\theta = |r| \times \sqrt{1-\cos^2\theta}$

$$= \sqrt{11} \times \sqrt{1-\left(\frac{6}{\sqrt{66}}\right)^2}$$

$$= \sqrt{5}$$

3 In this question you must show detailed reasoning.

Show that $\int_5^{\infty} (x-1)^{-\frac{3}{2}} \mathrm{d}x = 1$. **[5]**

$$\lim_{t\to\infty} \int_5^t (x-1)^{-\frac{3}{2}}\, dx$$

$$= \lim_{t\to\infty} \left[-2(x-1)^{-\frac{1}{2}}\right]_5^t$$

$$= \lim_{t\to\infty} \left(-2(t-1)^{-\frac{1}{2}} + 2(5-1)^{-\frac{1}{2}}\right)$$

$$= \lim_{t\to\infty} \left(-2(t-1)^{-\frac{1}{2}} + 1\right)$$

As $t \to \infty$, $\dfrac{-2}{\sqrt{t-1}} \to 0$

$\therefore \int_5^{\infty} (x-1)^{-\frac{3}{2}}\, dx = 1$ (as required)

4 A 2-D transformation T is a shear which leaves the y-axis invariant and which transforms the object point (2, 1) to the image point (2, 9). **A** is the matrix which represents the transformation T.

(a) Find **A**. **[3]**

(b) By considering the determinant of **A**, explain why the area of a shape is invariant under T. **[2]**

a) $A = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$

$A = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2k+1 \end{pmatrix} = \begin{pmatrix} 2 \\ 9 \end{pmatrix}$

$\therefore\ 2k+1 = 9$

$2k = 8$

$k = 4$

So $A = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$

b) $\det A = (1 \times 1) + (4 \times 0) = 1$

The determinant is the area scale factor so a determinant of 1 leaves the area unchanged.

5 A particle of mass 2 kg moves along the x-axis. At time t seconds the velocity of the particle is $v\,\mathrm{m\,s^{-1}}$.

The particle is subject to two forces.

- One acts in the positive x-direction with magnitude $\frac{1}{2}t\,\mathrm{N}$.
- One acts in the negative x-direction with magnitude $v\,\mathrm{N}$.

(a) Show that the motion of the particle can be modelled by the differential equation

$$\frac{dv}{dt}+\frac{1}{2}v=\frac{1}{4}t.$$ **[1]**

The particle is at rest when $t = 0$.

(b) Find v in terms of t. **[5]**

(c) Find the velocity of the particle when $t = 2$. **[1]**

When $t = 2$ the force acting in the **positive** x-direction is replaced by a constant force of magnitude $\frac{1}{2}\mathrm{N}$ in the same direction.

(d) Refine the differential equation given in part **(a)** to model the motion for $t \geqslant 2$. **[1]**

(e) Use the refined model from part **(d)** to find an exact expression for v in terms of t for $t \geqslant 2$. **[3]**

a) $F = ma \qquad \frac{dv}{dt} = a$

$\frac{1}{2}t - v = 2 \times \frac{dv}{dt}$

$\frac{1}{4}t - \frac{1}{2}v = \frac{dv}{dt}$

$\Rightarrow \frac{dv}{dt} + \frac{1}{2}v = \frac{1}{4}t$ (as required)

b) IF: $e^{\int \frac{1}{2}dt} = e^{\frac{1}{2}t}$

$e^{\frac{1}{2}t}\frac{dv}{dt} + \frac{1}{2}e^{\frac{1}{2}t}v = \frac{d}{dt}\left(ve^{\frac{1}{2}t}\right) = \frac{1}{4}te^{\frac{1}{2}t}$

$ve^{\frac{1}{2}t} = \frac{1}{2}te^{\frac{1}{2}t} - \frac{1}{2}\int e^{\frac{1}{2}t}dt$

$= \frac{1}{2}te^{\frac{1}{2}t} - e^{\frac{1}{2}t} + c$

When $t=0$, $v=0$

$0 = -1 + c$

$\therefore 1 = c$

$\therefore v = \frac{1}{2}t - 1 + e^{-\frac{1}{2}t}$

c) when $t=2$

$$v = \frac{1}{2}(2) - 1 + e^{-\frac{1}{2}(2)}$$

$$= e^{-1}$$

(so velocity $= e^{-1}\ ms^{-1}$)

d) Change to:

$$\frac{dv}{dt} + \frac{1}{2}v = \frac{1}{4}$$

e) $$\frac{d}{dt}\left(ve^{\frac{1}{2}t}\right) = \frac{1}{4}e^{\frac{1}{2}t}$$

$$ve^{\frac{1}{2}t} = \int \frac{1}{4}e^{\frac{1}{2}t}\,dt$$

$$ve^{\frac{1}{2}t} = \frac{1}{2}e^{\frac{1}{2}t} + c$$

when $t=2$, $v=e^{-1}$

$$e^{-1}e^{1} = \frac{1}{2}e^{1} + c$$

$$c = 1 - \frac{1}{2}e^{1}$$

$$\therefore v = \frac{1}{2} + \left(1 - \frac{1}{2}e\right)e^{-\frac{1}{2}t}$$

6 A is a fixed point on a smooth horizontal surface. A particle P is initially held at A and released from rest.

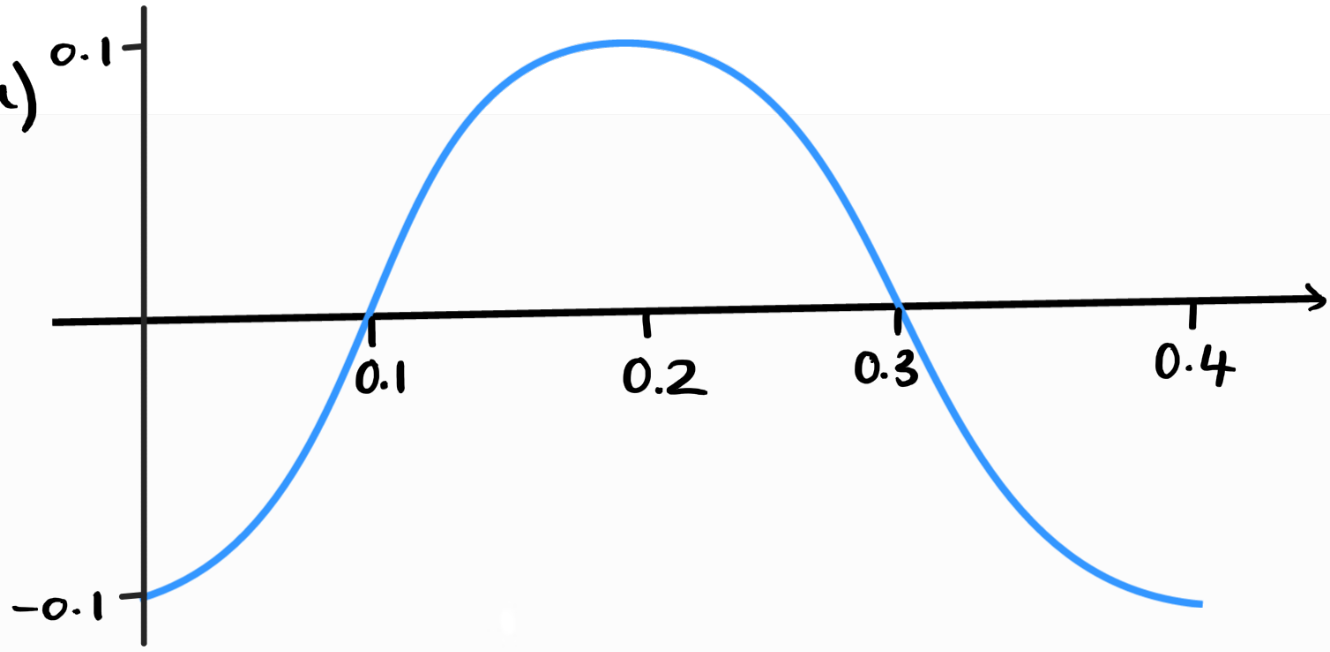
It subsequently performs simple harmonic motion in a straight line on the surface. After its release it is next at rest after 0.2 seconds at point B whose displacement is 0.2 m from A. The point M is halfway between A and B.

The displacement of P from M at time t seconds after release is denoted by x m.

(a) On the axes provided in the Printed Answer Booklet, sketch a graph of x against t for $0 \leqslant t \leqslant 0.4$. **[4]**

(b) Find the displacement of P from M at 0.75 seconds after release. **[2]**

a)



b) So $x = \pm 0.1\cos(5\pi t)$

when $t = 0.75$

$x = \dfrac{\sqrt{2}}{20}$

$= -0.0707$ (3sf)

7 In an Argand diagram the points representing the numbers $2 + 3\text{i}$ and $1 - \text{i}$ are two adjacent vertices of a square, S.

(a) Find the area of S. **[3]**

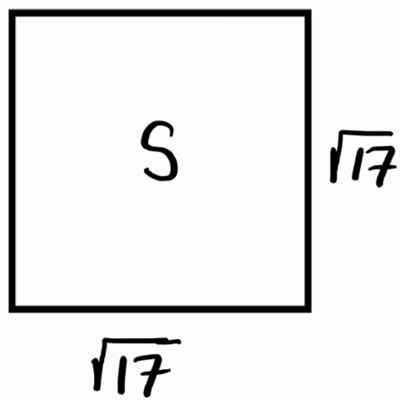
(b) Find all the possible pairs of numbers represented by the other two vertices of S. **[4]**

a) $(2+3i)-(1-i) = \pm(1+4i)$

$|1+4i| = \sqrt{1^2+4^2}$

$= \sqrt{17}$

$\sqrt{17} \times \sqrt{17} = 17$



b) $\pm i \times \pm(1+4i)$

$2+3i \pm i(1+4i)$ and $(1-i) \pm i(1+4i)$

So vertices are at -3 and $-2+4i$

or at $5-2i$ and $6+2i$

8 In this question you must show detailed reasoning.

(a) By writing $\sin\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$ show that

$$\sin^6\theta = \frac{1}{32}(10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta).$$ **[5]**

(b) Hence show that $\sin\frac{1}{8}\pi = \frac{1}{2}\sqrt[6]{20 - 14\sqrt{2}}$. **[3]**

a) $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

$\sin^6\theta = \left(\dfrac{e^{i\theta} - e^{-i\theta}}{2i}\right)^6 = -\dfrac{1}{64}\left(e^{i\theta} - e^{-i\theta}\right)^6$

$(e^{i\theta} - e^{-i\theta})^6 = e^{6i\theta} - 6e^{4i\theta} + 15e^{2i\theta} - 20 + 15e^{-2i\theta} - 6e^{-4i\theta} + e^{-6i}$

$= e^{6i\theta} + e^{-6i\theta} - 6(e^{4i\theta} + e^{-4i\theta}) + 15(e^{2i\theta} + e^{-2i\theta}) - 20$

$= 2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20$

$\therefore \sin^6\theta = -\dfrac{1}{64}(2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20)$

$= \dfrac{1}{32}(10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta)$

b) when $\theta = \dfrac{\pi}{8}$:

$\cos 2\theta = \cos\dfrac{2\pi}{8} = \dfrac{\sqrt{2}}{2}$, $\cos 4\theta = 0$, $\cos 6\theta = \dfrac{-\sqrt{2}}{2}$

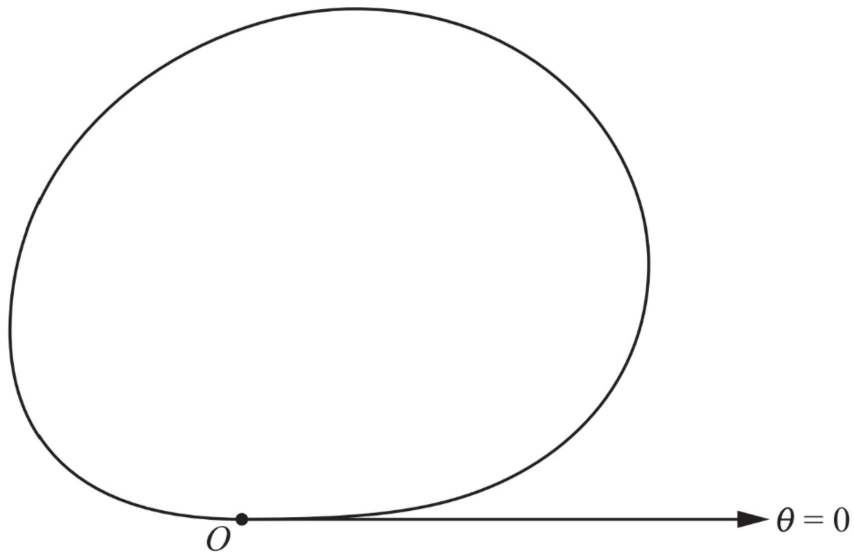
$\sin^6\dfrac{\pi}{8} = \dfrac{1}{32}\left(10 - \dfrac{15\sqrt{2}}{2} + (6\times 0) - \left(-\dfrac{\sqrt{2}}{2}\right)\right)$

$$\sin\frac{\pi}{8} = \sqrt[6]{\frac{1}{64}\left(20-15\sqrt{2}+\sqrt{2}\right)}$$

$$\sin\frac{\pi}{8} = \frac{1}{2}\sqrt[6]{20-14\sqrt{2}} \quad \text{(as required)}$$

9 In this question you must show detailed reasoning.

The diagram below shows the curve $r=\sqrt{\sin\theta}\,e^{\frac{1}{3}\cos\theta}$ for $0 \leqslant \theta \leqslant \pi$.



(a) Find the exact area enclosed by the curve. **[4]**

(b) Show that the greatest value of r on the curve is $\sqrt{\frac{\sqrt{3}}{2}e^{\frac{1}{6}}}$. **[7]**

a) $\frac{1}{2}\int\left(\sqrt{\sin\theta}\,e^{\frac{1}{3}\cos\theta}\right)^2 d\theta$

$$A = \frac{1}{2}\int_0^{\pi} \sin\theta\, e^{\frac{2}{3}\cos\theta}\, d\theta$$

$$= \frac{1}{2}\left[-\frac{3}{2}e^{\frac{2}{3}\cos\theta}\right]_0^{\pi}$$

$$= \frac{3}{4}\left(e^{\frac{2}{3}} - e^{-\frac{2}{3}}\right)$$

b) $\frac{dr}{d\theta} = \frac{1}{2}\cos\theta(\sin\theta)^{-\frac{1}{2}}e^{\frac{1}{3}\cos\theta} + (\sin\theta)^{\frac{1}{2}}\left(-\frac{1}{3}\sin\theta\right)e^{\frac{1}{3}\cos\theta}$

$\frac{dr}{d\theta} = \frac{1}{6}(\sin\theta)^{-\frac{1}{2}}e^{\frac{1}{3}\cos\theta}(3\cos\theta - 2\sin^2\theta)$

when $\frac{dr}{d\theta} = 0$

$0 = \frac{1}{6}(\sin\theta)^{-\frac{1}{2}}e^{\frac{1}{3}\cos\theta}(3\cos\theta - 2\sin^2\theta)$

$\Rightarrow 3\cos\theta - 2\sin^2\theta = 0$

$3\cos\theta - 2(1-\cos^2\theta) = 0$

$2\cos^2\theta + 3\cos\theta - 2 = 0$

$(2\cos\theta - 1)(\cos\theta + 2) = 0$

$\cos\theta = \frac{1}{2}, -2$

$\cos\theta \neq -2$

$\Rightarrow \sin\theta = \frac{\sqrt{3}}{2} \Rightarrow r = \sqrt{\frac{\sqrt{3}}{2}}\, e^{\frac{1}{3}\times\frac{1}{2}} = \sqrt{\frac{\sqrt{3}}{2}}\, e^{\frac{1}{6}}$

(as required)

10 **(a)** Use differentiation to find the first two non-zero terms of the Maclaurin expansion of $\ln\left(\frac{1}{2} + \cos x\right)$. **[4]**

(b) By considering the root of the equation $\ln\left(\frac{1}{2} + \cos x\right) = 0$ deduce that $\pi \approx 3\sqrt{3\ln\left(\frac{3}{2}\right)}$. **[3]**

a) $f(0) = \ln\left(\frac{1}{2} + \cos 0\right) = \ln\left(\frac{3}{2}\right)$

$\frac{d}{dx}\left(\ln\left(\frac{1}{2} + \cos x\right)\right) = \frac{-\sin x}{\frac{1}{2} + \cos x} \Rightarrow f'(0) = 0$

$$\frac{d^2 \ln\left(\frac{1}{2}+\cos x\right)}{dx^2} = \frac{-\cos x\left(\frac{1}{2}+\cos x\right)+\sin x(-\sin x)}{\left(\frac{1}{2}+\cos x\right)^2}$$

$$\Rightarrow f''(0) = -\frac{2}{3}$$

$$\ln\left(\frac{1}{2}+\cos x\right) = \ln\left(\frac{3}{2}\right) - \frac{x^2}{3} + \dots$$

b) $\ln\left(\frac{1}{2}+\cos x\right) = 0$

when $x = \frac{\pi}{3}$

$$\ln\left(\frac{3}{2}\right) - \frac{\left(\frac{\pi}{3}\right)^2}{3} \approx 0$$

$$\ln\frac{3}{2} - \frac{\pi^2}{27} \approx 0$$

$$\therefore \pi \approx \sqrt{27\ln\frac{3}{2}}$$

$$= 3\sqrt{3\ln\frac{3}{2}} \quad \text{(as required)}$$