OCR
Oxford Cambridge and RSA

MODEL SOLUTIONS

Accredited

A Level Further Mathematics A
Y540 Pure Core 1
Sample Question Paper

Date – Morning/Afternoon
Time allowed: 1 hour 30 minutes

OCR supplied materials:
- Printed Answer Booklet
- Formulae A Level Further Mathematics A

You must have:
- Printed Answer Booklet
- Formulae A Level Further Mathematics A
- Scientific or graphical calculator

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INSTRUCTIONS

- Use black ink. HB pencil may be used for graphs and diagrams only.
- Complete the boxes provided on the Printed Answer Booklet with your name, centre number and candidate number.
- Answer **all** the questions.
- **Write your answer to each question in the space provided in the Printed Answer Booklet.**
- Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- Do **not** write in the bar codes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The acceleration due to gravity is denoted by $g\,\text{m}\,\text{s}^{-2}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION

- The total number of marks for this paper is **75**.
- The marks for each question are shown in brackets **[ ]**.
- **You are reminded of the need for clear presentation in your answers.**
- The Printed Answer Booklet consists of **12** pages. The Question Paper consists of **8** pages.

Answer **all** the questions.

1 Show that $\dfrac{5}{2-4\mathrm{i}} = \frac{1}{2} + \mathrm{i}$. **[2]**

$$= \frac{5}{2-4i} \times \frac{2+4i}{2+4i} = \frac{10+20i}{4+16} = \frac{10(1+2i)}{20}$$

$$= \left(\tfrac{1}{2} + i\right) \text{ shown.}$$

2 **In this question you must show detailed reasoning.**

The equation $\mathrm{f}(x) = 0$, where $\mathrm{f}(x) = x^4 + 2x^3 + 2x^2 + 26x + 169$, has a root $x = 2 + 3\mathrm{i}$.

(i) Express $\mathrm{f}(x)$ as a product of two quadratic factors. **[4]**

$x = 2+3i$ is a root, $\therefore x^* = 2-3i$ is also a root.

$\sum x = 2+3i+2-3i = 4 \qquad xx^* = 13$

$\therefore$ Quadratic factor $= x^2 - 4x + 13$

Other quadratic factor: $(x^2 + ax + 13)$ $\{$as $13 \times 13 = 169\}$

$\Rightarrow f(x) = (x^2 - 4x + 13)(x^2 + ax + 13)$

$\Rightarrow 13ax - 13 \times 4x = 26 \Rightarrow a = 6$

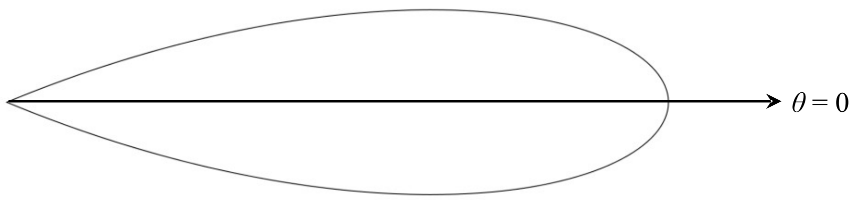
$\therefore f(x) = (x^2 - 4x + 13)(x^2 + 6x + 13)$

(ii) Hence write down all the roots of the equation $\mathrm{f}(x) = 0$. **[1]**

Roots: $2 \pm 3i$, $-3 \pm 2i$

3 **In this question you must show detailed reasoning.**

The diagram below shows the curve $r = 2\cos 4\theta$ for $-k\pi \le \theta \le k\pi$ where k is a constant to be determined.



Calculate the exact area enclosed by the curve. **[6]**

$r = 0$ when $4\theta = \pm\frac{\pi}{2} \Rightarrow k = \frac{1}{8}$

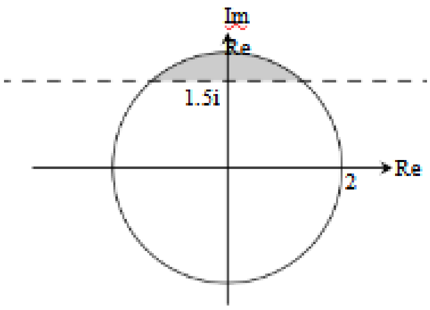
$$\text{Area} = \frac{1}{2}\int_{-\pi/8}^{\pi/8} r^2 \, d\theta$$

$= \frac{1}{2}\int_{-\pi/8}^{\pi/8} 4\cos^2 4\theta \, d\theta$ $\quad \{4\cos^2\theta = 2(\cos 8\theta + 1)\}$

$= \frac{1}{2}\int_{-\pi/8}^{\pi/8} 2(\cos 8\theta + 1)\, d\theta = \int_{-\pi/8}^{\pi/8} \cos 8\theta + 1 \, d\theta$

$= \left[\theta + \frac{1}{8}\sin 8\theta\right]_{-\pi/8}^{\pi/8} = \left(\frac{\pi}{8}+0\right) - \left(-\frac{\pi}{8}+0\right) = \frac{\pi}{4}$

4 Draw the region in an Argand diagram for which $|z| \le 2$ and $|z| > |z - 3\mathrm{i}|$. **[3]**



5 **(i)** Show that $\frac{\mathrm{d}}{\mathrm{d}x}\left(\sinh^{-1}(2x)\right) = \frac{2}{\sqrt{4x^2+1}}$. **[2]**

$\frac{d}{dx}\left(\sinh^{-1}(2x)\right) = \frac{1}{\sqrt{x^2+\left(\frac{1}{2}\right)^2}} = \frac{1}{\sqrt{x^2+1/4}} \times \frac{2}{2}$

$= \frac{2}{\sqrt{4x^2+1}}$ shown.

(ii) Find $\int \frac{1}{\sqrt{2-2x+x^2}}\,\mathrm{d}x$. **[3]**

$\Rightarrow \int \frac{1}{\sqrt{(x-1)^2+1}}\,dx$ $\quad 2-2x+x^2 = (x-1)^2+1$

$= \sinh^{-1}(x-1) + c$

6 The equation $x^3 + 2x^2 + x + 3 = 0$ has roots α, β and γ.

The equation $x^3 + px^2 + qx + r = 0$ has roots $\alpha\beta, \beta\gamma$ and $\gamma\alpha$.

Find the values of p, q and r. **[5]**

$\Sigma\alpha = -2 = -\frac{b}{a}$ $\quad \Sigma\alpha\beta = 1 = \frac{c}{a}$ $\quad \Sigma\alpha\beta\gamma = -3 = -\frac{d}{a}$

$-p = \alpha\beta + \alpha\gamma + \beta\gamma = \Sigma\alpha\beta = 1 \quad \therefore -p = 1 \Rightarrow \underline{p = -1}$

$q = \alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2 = \alpha\beta\gamma(\alpha + \beta + \gamma)$

$= -3(-2) = 6 = q \quad \therefore \underline{q = 6}$

$-r = \alpha^2\beta^2\gamma^2 = (\alpha\beta\gamma)^2 = 9 \quad \therefore \underline{\underline{r = -9}}$

$\therefore \underline{p = -1}, \ \underline{q = 6}, \ \underline{r = -9}$

7 The lines l_1 and l_2 have equations $\frac{x-3}{1} = \frac{y-5}{2} = \frac{z+2}{-3}$ and $\frac{x-4}{2} = \frac{y+2}{-1} = \frac{z-7}{4}$.

(i) Find the shortest distance between l_1 and l_2. **[5]**

$l_1 = \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \qquad l_2 = \begin{pmatrix} 4 \\ -2 \\ 7 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$

$\text{Distance} = \frac{|\vec{AP} \cdot (\lambda \times \mu)|}{|\lambda \times \mu|} \qquad \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ -10 \\ -5 \end{pmatrix}$

$\vec{AP} = \begin{pmatrix} 1 \\ -7 \\ 5 \end{pmatrix} \Rightarrow \text{Distance} = \frac{\begin{pmatrix} 1 \\ -7 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -10 \\ -5 \end{pmatrix}}{\sqrt{2 \times 5^2 + 10^2}} = \underline{\underline{\sqrt{6}}}$

(ii) Find a cartesian equation of the plane which contains l_1 and is parallel to l_2. **[2]**

$\begin{pmatrix} 5 \\ -10 \\ -5 \end{pmatrix}$ is the vector parallel to both l_1 and l_2;

$\therefore \text{Eqn.} = 5x - 10y - 5z = d$

Subs point $(3, 5, -2) \Rightarrow 15 - 50 + 10 = d$

$\Rightarrow d = -25$

$\therefore 5x - by - 5z = -25 \Rightarrow \underline{\underline{x - 2y - z = -5}}$

8 **(i)** Find the solution to the following simultaneous equations.

$$\begin{aligned} x + \quad y + \quad z &= 3 \\ 2x + \quad 4y + \quad 5z &= 9 \\ 7x + \quad 11y + \quad 12z &= 20 \end{aligned}$$

[2]

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 5 \\ 7 & 11 & 12 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \\ 20 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 5 \\ 7 & 11 & 12 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 9 \\ 20 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 7 & 1 & -1 \\ -11 & -5 & 3 \\ 6 & 4 & -2 \end{pmatrix} \begin{pmatrix} 3 \\ 9 \\ 20 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 10 \\ -18 \\ 14 \end{pmatrix} \Rightarrow x, y, z = \underline{\underline{(5, -9, 7)}}$$

(ii) Determine the values of p and k for which there are an infinity of solutions to the following simultaneous equations.

$$\begin{aligned} x + \quad y + \quad z &= 3 \quad (1) \\ 2x + \quad 4y + \quad 5z &= 9 \quad (2) \\ 7x + \quad 11y + \quad pz &= k \quad (3) \end{aligned}$$

[6]

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 5 \\ 7 & 11 & p \end{pmatrix} \qquad \text{Det } M = 4p + 35 + 22 - 28 - 2p - 55$$

M is singular, $\therefore$ Det M $= 2p - 26 = 0$

$\Rightarrow \underline{\underline{p = 13}}$

If ∞ no. of solutions $\Rightarrow$ sheaf $\therefore$ consistent eqns.

Solving (1), (2), (3): $\qquad 2(1) - (2)$

$$\begin{aligned} \cancel{2x} + 2y + 2z &= 6 \\ \cancel{2x} + 4y + 5z &= 9 \\ (-) \quad (-) \quad (-) \quad &(-) \\ \hline -2y - 3z &= -3 \\ 2y + 3z &= 3 \end{aligned}$$

$7(1) - 3 \Rightarrow$

$$\begin{aligned} \cancel{7x} + 7y + 7z &= 21 \\ \cancel{7x} + 11y + 13z &= k \\ (-) \quad (-) \quad (-) \quad &(-) \\ \hline -4y - 6z &= 21 - k \end{aligned}$$

For consistency:

$\Rightarrow 21 - k = -2 \times 3$

$\Rightarrow K = \underline{\underline{27}}$

9 Prove by induction that, for all positive integers n,

$$\sum_{r=1}^{n} \frac{5-4r}{5^r} = \frac{n}{5^n}.$$

[5]

Proving true for $n=1$ $\quad \sum_{r=1}^{1} \frac{5-4}{5} = \frac{1}{5}$

$RHS = \frac{1}{5^1} = \frac{1}{5}$ $\therefore$ true for $\underline{n=1}$

Assuming true for $n=k$, i.e. $\sum_{r=1}^{k} \frac{5-4r}{5^r} = \frac{k}{5^k}$

Checking for $n=k+1$:

$$\sum_{r=1}^{k+1} = \sum_{r=1}^{k} + \sum_{r=k+1}^{k+1}$$

$$\Rightarrow \sum_{r=1}^{k+1} \frac{5-4r}{5^r} = \frac{k}{5^k} + \frac{5-4(k+1)}{5^{k+1}}$$

$$= \frac{k}{5^k} + \frac{1-4k}{5^{k+1}}$$

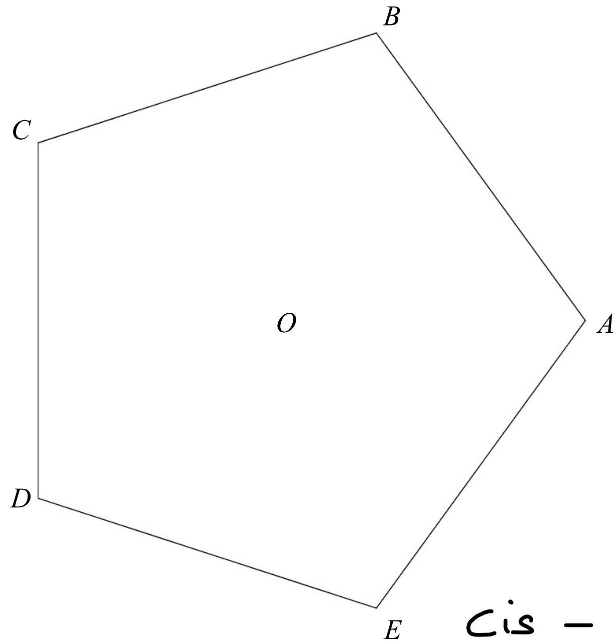
$$= \frac{5k+1-4k}{5^{k+1}}$$

$$= \frac{k+1}{5^{k+1}} \quad \text{as } \underline{\text{needed}}$$

$\therefore$ If true for $n=k$, it's true for $n=k+1$ and because it's true for $n=1$, it must be true for $n \in \mathbb{Z}^+$ by mathematical induction.

10 The Argand diagram below shows the origin O and pentagon $ABCDE$, where A, B, C, D and E are the points that represent the complex numbers a, b, c, d and e, and where a is a positive real number.

You are given that these five complex numbers are the roots of the equation $z^5 - a^5 = 0$.



Cis – 'Complex no.'

(i) Justify each of the following statements.

(a) A, B, C, D and E lie on a circle with centre O. **[1]**

All roots satisfy $|z^5| = a^5 \Rightarrow |z| = a$

(b) $ABCDE$ is a regular pentagon. **[2]**

All points are distance a from O. Each root is $a\,\mathrm{Cis}\left(\frac{2k\pi}{5}\right)$, hence spaced at intervals of $\frac{2\pi}{5}$ around circle.

(c) $b \times e^{\frac{2i\pi}{5}} = c$ **[1]**

$$\arg\left(b \times e^{\frac{2\pi}{5}i}\right) = \arg(b) + \arg\left(e^{\frac{2}{5}\pi i}\right)$$
$$= \tfrac{4}{5}\pi = \arg(c)$$

(d) $b^* = e$ **[1]**

$$b^* = \mathrm{Cis}\left(-\frac{2\pi}{5}\right) = \mathrm{Cis}\left(\frac{8\pi}{5}\right) = e$$

(e) $a + b + c + d + e = 0$ **[2]**

a, b, c, d, e are roots of $z^5 - a^5 = 0$ $\therefore$ Sum of $a, b, c, d, e = -\frac{0}{1} = 0$.

(ii) The midpoints of sides AB, BC, CD, DE and EA represent the complex numbers p, q, r, s and t. Determine a polynomial equation, with real coefficients, that has roots p, q, r, s and t. **[3]**

All roots have mag. $a\cos\frac{\pi}{5}$

All roots are spaced at angles of $\frac{2\pi}{5}$.

r is negative real root $\Rightarrow z^5 + \left(a\cos\frac{\pi}{5}\right)^5 = 0$.

11 A company is required to weigh any goods before exporting them overseas. When a crate is placed on a set of weighing scales, the mass displayed takes time to settle down to its final value.

The company wishes to model the mass, m kg, which is displayed t seconds after a crate X is placed on the scales.

For the displayed mass it is assumed that the rate of change of the quantity $\left(0.5\frac{\mathrm{d}m}{\mathrm{d}t}+m\right)$ with respect to time is proportional to $(80-m)$.

(i) Show that $\frac{\mathrm{d}^2m}{\mathrm{d}t^2}+2\frac{\mathrm{d}m}{\mathrm{d}t}+2km=160k$, where k is a real constant. **[2]**

$$\frac{d}{dt}\left(0.5\frac{dm}{dt}+m\right) = k(80-m)$$

$$0.5\frac{d^2m}{dt^2}+\frac{dm}{dt} = 80k - km \Rightarrow \frac{d^2m}{dt^2}+2\frac{dm}{dt} = 160k - 2km$$

$$\Rightarrow \frac{d^2m}{dt^2}+2\frac{dm}{dt}+2km = 160k \quad \text{shown.}$$

It is given that the complementary function for the differential equation in part **(i)** is $e^{\lambda t}(A\cos 2t + B\sin 2t)$, where A and B are arbitrary constants.

(ii) Show that $k=\frac{5}{2}$ and state the value of the constant λ. **[4]**

A.E. $\therefore n^2 + 2n + 2k = 0 \Rightarrow n = \dfrac{-2 \pm \sqrt{4-8k}}{2}$

$= -1 \pm \sqrt{1-2k}$

as $A\cos 2t + B\sin 2t \Rightarrow 1-2k < 0$

$\Rightarrow \sqrt{2k-1} = 2 \Rightarrow k = 5/2 \Rightarrow \lambda = -1$

When X is initially placed on the scales the displayed mass is zero and the rate of increase of the displayed mass is $160\,\mathrm{kg\,s^{-1}}$.

(iii) Find m in terms of t. **[7]**

$$\frac{d^2m}{dt^2}+2\frac{dm}{dt}+5m = 400$$

let $m = \mu \Rightarrow 5\mu = 400 \Rightarrow \mu = 80$

as P.I. $\frac{d^2m}{dt^2} = \frac{dm}{dt} = 0$

$m = e^{-t}(A\cos 2t + B\sin 2t) + 80$

at $t = 0$ $m = 0$ $\therefore A = -80$

$\frac{dm}{dt} = -e^{-t}(-80\cos 2t + B\sin 2t) + e^{-t}(-2(-80)\sin 2t + 2B\cos 2t)$

@$t = 0$ $\frac{dm}{dt} = 160 \therefore -(-80) + 2B = 160 \Rightarrow B = \underline{\underline{40}}$

$\Rightarrow m = e^{-t}(40\sin 2t - 80\cos 2t) + 80$

(iv) Describe the long term behaviour of *m*. **[1]**

$t \to \infty$ $m \to 80$

(v) With reference to your answer to part **(iv)**, comment on a limitation of the model. **[1]**

Mass of crate is shown only after ∞ time.

(vi) **(a)** Find the value of *m* that corresponds to the stationary point on the curve $m = \mathrm{f}(t)$ with the smallest positive value of *t*. **[2]**

$m = e^{-t}(40\sin 2t - 80\cos 2t) + 80$

$\frac{dm}{dt} = e^{-t}(160\cos 2t + 120\sin 2t) = 0$

$\Rightarrow t = 1.107... \Rightarrow m = \underline{106}$

(b) Interpret this value of *m* in the context of the model. **[1]**

The above is the maximum mass displayed.

(vii) Adapt the differential equation $\frac{d^2m}{dt^2} + 2\frac{dm}{dt} + 5m = 400$ to model the mass displayed *t* seconds after a crate Y, of mass 100 kg, is placed on the scales. **[1]**

$\frac{d^2m}{dt^2} + 2\frac{dm}{dt} + 5m = 500$

END OF QUESTION PAPER