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MODEL ANSWERS

Monday 05 October 2020 – Afternoon

A Level Further Mathematics A

Y540/01 Pure Core 1

Time allowed: 1 hour 30 minutes

You must have:
- the Printed Answer Booklet
- the Formulae Booklet for A Level Further Mathematics A
- a scientific or graphical calculator

INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question.
- The acceleration due to gravity is denoted by $g\,\text{m}\,\text{s}^{-2}$. When a numerical value is needed use $g = 9.8$ unless a different value is specified in the question.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION
- The total mark for this paper is **75**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **8** pages.

ADVICE
- Read each question carefully before you start your answer.

1 Find the mean value of $f(x) = x^2 + 6x$ over the interval $[0, 3]$. **[2]**

$$\text{Mean Value} = \frac{1}{3-0}\int_0^3 f(x)\,dx$$

$$= \frac{1}{3}\int_0^3 x^2 + 6x\,dx$$

$$= \frac{1}{3}\left[\frac{x^3}{3} + \frac{6x^2}{2}\right]_0^3$$

$$= \frac{1}{3}\left[\frac{3^3}{3} + 3(3)^2\right]$$

$$= 12$$

$$\boxed{\therefore \text{Mean Value} = 12}$$

2 Find an expression for $1\times2^2 + 2\times3^2 + 3\times4^2 + \ldots + n(n+1)^2$ in terms of n. Give your answer in fully factorised form. **[3]**

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$$

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

$$\sum_{r=1}^{n} r(r+1)^2 = \sum_{r=1}^{n} r^3 + 2r^2 + r$$

$$= \sum_{r=1}^{n} r^3 + 2\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r$$

$$= \frac{1}{4}n^2(n+1)^2 + \frac{2}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1)$$

$$= \frac{1}{12}n(n+1)\left[3n(n+1) + 4(2n+1) + 6\right]$$

$$= \frac{1}{12}n(n+1)\left[3n^2 + 3n + 8n + 4 + 6\right]$$

$$= \frac{1}{12}n(n+1)(3n^2 + 11n + 10)$$

$$= \boxed{\frac{1}{12}n(n+1)(n+2)(3n+5)}$$

3 You are given the matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$.

(a) Find $\mathbf{A}^4$. **[1]**

(b) Describe the transformation that $\mathbf{A}$ represents. **[2]**

The matrix $\mathbf{B}$ represents a reflection in the plane $x = 0$.

(c) Write down the matrix $\mathbf{B}$. **[1]**

The point P has coordinates $(2, 3, 4)$. The point P' is the image of P under the transformation represented by $\mathbf{B}$.

(d) Find the coordinates of P'. **[1]**

(a.) $$\underline{A} \cdot \underline{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \underline{A}^2$$

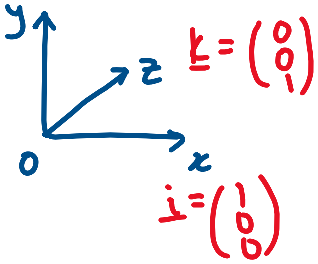
$$\underline{A}^4 = \underline{A}^2 \cdot \underline{A}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \underline{I}$$

$$\boxed{\therefore \underline{A}^4 = \underline{I}}$$

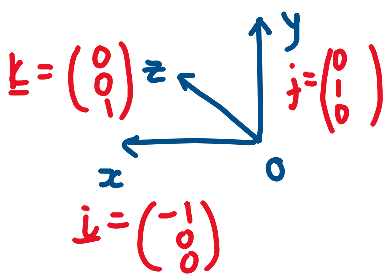
(b.) Matrix $\underline{A}$ represents 90° clockwise rotation about x-axis.

(c.) $$\boxed{\underline{B} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

$\underline{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$



Reflection in $x = 0$:



(d.) $P = (2, 3, 4)$

$$P' = \underline{B} \cdot \underline{P} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix}$$

$$\boxed{\therefore P' = (-2, 3, 4)}$$

4 In this question you must show detailed reasoning.

(a) Determine the square roots of 25i in the form $re^{i\theta}$, where $0 \leqslant \theta < 2\pi$. **[3]**

(b) Illustrate the number 25i and its square roots on an Argand diagram. **[1]**

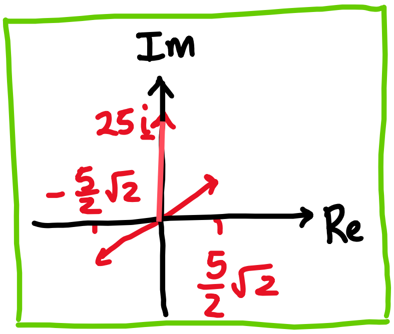
(a.) $r = 25$

$\theta = \frac{\pi}{2}$

$\therefore 25i = 25e^{\frac{\pi}{2}i}$

$\boxed{\sqrt{25i} = 5e^{\frac{\pi}{4}i} \ \& \ 5e^{\frac{5\pi}{4}i}}$

(b.)



5 By expanding $\left(z^2+\frac{1}{z^2}\right)^3$, where $z=e^{i\theta}$, show that $4\cos^3 2\theta = \cos 6\theta + 3\cos 2\theta$. **[5]**

$$\left(z^2+\frac{1}{z^2}\right)^3 = \left(z^2+\frac{1}{z^2}\right)\left(z^4+2+\frac{1}{z^4}\right)$$

$$= z^6 + 2z^2 + \frac{1}{z^2} + z^2 + \frac{2}{z^2} + \frac{1}{z^6}$$

$$= z^6 + 3z^2 + \frac{3}{z^2} + \frac{1}{z^6}$$

$$\left(z^2+\frac{1}{z^2}\right) = 2\cos 2\theta \Rightarrow \left(z^2+\frac{1}{z^2}\right)^3 = 8\cos^3 2\theta$$

Using De Moivre's

$$3\left(z^2+\frac{1}{z^2}\right) = 6\cos 2\theta$$

$$z^6 + \frac{1}{z^6} = 2\cos 6\theta$$

$$\left(z^2+\frac{1}{z^2}\right)^3 = \left(z^6+\frac{1}{z^6}\right) + 3\left(z^2+\frac{1}{z^2}\right)$$

$$8\cos^3 2\theta = 2\cos 6\theta + 6\cos 2\theta$$

$$\boxed{\therefore 4\cos^3 2\theta = \cos 6\theta + 3\cos 2\theta}$$

6 The equations of two non-intersecting lines, l_1 and l_2, are

$$l_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \qquad l_2 : \mathbf{r} = \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}.$$

Find the shortest distance between lines l_1 and l_2. **[5]**

$$\underline{n} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} = \underline{i}\begin{vmatrix} 1 & -2 \\ -1 & 4 \end{vmatrix} - \underline{j}\begin{vmatrix} 2 & -2 \\ 1 & 4 \end{vmatrix} + \underline{k}\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}$$

$$\therefore \underline{n} = 2\underline{i} - 10\underline{j} - 3\underline{k} = \begin{pmatrix} 2 \\ -10 \\ -3 \end{pmatrix}$$

$$|\underline{n}| = \sqrt{2^2 + (-10)^2 + (-3)^2} = \sqrt{113}$$

$$\underline{b} - \underline{a} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

$$d = \frac{\left| \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -10 \\ -3 \end{pmatrix} \right|}{|\underline{n}|} = \frac{|-1(2) + 0(-10) + 2(-3)|}{\sqrt{113}} = \frac{8}{\sqrt{113}}$$

$$\boxed{\therefore d \approx 0.753 \text{ (3sf)}}$$

7 Prove by induction that the sum of the cubes of three consecutive positive integers is divisible by 9. **[5]**

① Base case:

$1^3 + 2^3 + 3^3 = 36 = 4 \times 9$ $\therefore$ True for base case.

② Consider sum: $f(r) = r^3 + (r+1)^3 + (r+2)^3$

Assume that $f(r) = 9k$ for some $k \in \mathbb{Z}$.

③ Consider $f(r+1)$:

$$f(r+1) = f(r) + (r+3)^3 - r^3$$
$$= f(r) + (r+3)(r^2 + 6r + 9) - r^3$$
$$= f(r) + \cancel{r^3} + 6r^2 + 9r + 3r^2 + 18r + 27 - \cancel{r^3}$$
$$= f(r) + 9r^2 + 27r + 27$$
$$= 9k + 9(r^2 + 3r + 3)$$
$$= 9k' \text{ for some } k' \in \mathbb{Z}$$

④ $\therefore$ If true for r, then true for $r+1$. But it is also true for $r=1$, so true for all integers r.

8 **(a)** Using exponentials, show that $\cosh 2u \equiv 2\sinh^2 u + 1$. **[2]**

(b) By differentiating both sides of the identity in part **(a)** with respect to u, show that

$\sinh 2u \equiv 2\sinh u \cosh u$. **[1]**

(c) Use the substitution $x = \sinh^2 u$ to find $\int \sqrt{\frac{x}{x+1}}\,dx$. Give your answer in the form $a\sinh^{-1} b\sqrt{x} + f(x)$ where a and b are integers and $f(x)$ is a function to be determined. **[5]**

(d) Hence determine the exact area of the region between the curve $y = \sqrt{\frac{x}{x+1}}$, the x-axis, the line $x = 1$ and the line $x = 2$. Give your answer in the form $p + q\ln r$ where p, q and r are numbers to be determined. **[2]**

(a.) $2\sinh^2 u + 1 \equiv 2\left(\frac{e^u - e^{-u}}{2}\right)^2 + 1$

$= \frac{e^{2u} - \cancel{2} + e^{-2u}}{2} + \frac{\cancel{2}}{2}$

$= \frac{e^{2u} + e^{-2u}}{2}$

$\equiv \cosh 2u$

$\boxed{\therefore \cosh 2u \equiv 2\sinh^2 u + 1}$

(b.) $\frac{d}{du}[\cosh 2u] \equiv 2\sinh 2u$

$\frac{d}{du}[2\sinh^2 u + 1] \equiv 4\sinh u \cosh u$

$2\sinh 2u \equiv 4\sinh u \cosh u$

$\boxed{\therefore \sinh 2u \equiv 2\sinh u \cosh u}$

Remember

$\cosh 2u = \cosh^2 u + \sinh^2 u$

$\therefore \cosh 2u = 2\cosh^2 u - 1$

OR

$\cosh 2u = 1 + 2\sinh^2 u$

(c.) $x = \sinh^2 u \Rightarrow \frac{dx}{du} = 2\sinh u \cosh u$

$\therefore dx = 2\sinh u \cosh u \, du$

$$\int \sqrt{\frac{x}{x+1}} \, dx = \int \sqrt{\frac{\sinh^2 u}{\sinh^2 u + 1}} \times 2\sinh u \cosh u \, du$$

$$= 2\int \frac{\sinh u}{\cancel{\cosh u}} \times \sinh u \cancel{\cosh u} \, du$$

Since $\cosh^2 u = \sinh^2 u + 1$

$$= 2\int \sinh^2 u \, du$$

$$= \int \cosh 2u - 1 \, du$$

Since $2\sinh^2 u = \cos 2u - 1$

$$= \frac{1}{2}\sinh 2u - u + c$$

$$= \sinh u \cosh u - u + c$$

$$= \sqrt{x(1+x)} - \sinh^{-1}\sqrt{x} + c$$

$$\boxed{\therefore f(x) = \sqrt{x(1+x)} + c, \; a = -1, \; b = 1}$$

(d.) $\text{Area} = \left[\sqrt{x(1+x)} - \sinh^{-1}\sqrt{x}\right]_1^2$

$\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$

$$= \left(\sqrt{6} - \ln(\sqrt{2} + \sqrt{3})\right) - \left(\sqrt{2} - \ln(1+\sqrt{2})\right)$$

$$= \sqrt{6} - \sqrt{2} + \ln\left(\frac{1+\sqrt{2}}{\sqrt{2}+\sqrt{3}}\right)$$

$$\boxed{\therefore p = \sqrt{6} - \sqrt{2}, \; q = 1, \; r = \frac{1+\sqrt{2}}{\sqrt{2}+\sqrt{3}}}$$

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9 You are given that the cubic equation $2x^3 + px^2 + qx - 3 = 0$, where p and q are real numbers, has a complex root $\alpha = 1 + \mathrm{i}\sqrt{2}$.

(a) Write down a second complex root, β. **[1]**

(b) Determine the third root, γ. **[2]**

(c) Find the value of p and the value of q. **[2]**

(d) Show that if n is an integer then $\alpha^n + \beta^n + \gamma^n = 2 \times 3^{\frac{1}{2}n} \times \cos n\theta + \frac{1}{2^n}$ where $\tan\theta = \sqrt{2}$. **[4]**

(a.) $\boxed{\beta = 1 - i\sqrt{2}}$

(b.) $\alpha\beta\gamma = -\left(\frac{-3}{2}\right) = \frac{3}{2}$ product of roots

$\alpha\beta = (1+i\sqrt{2})(1-i\sqrt{2}) = 1 + i\sqrt{2} - i\sqrt{2} - 2i^2 = 1 + 2 = 3$

$\therefore \gamma = \frac{3}{2} \div \alpha\beta = \frac{3}{2} \div 3 = \frac{1}{2}$ $\boxed{\therefore \gamma = \frac{1}{2}}$

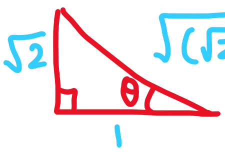
(c.) $(x - (1+i\sqrt{2}))(x - (1 - i\sqrt{2}))(2x-1) = 0$

$(x^2 - 2x + 3)(2x-1) = 0$

$2x^3 - x^2 - 4x^2 + 2x + 6x - 3 = 0$

$2x^3 - 5x^2 + 8x - 3 = 0$ $\boxed{\therefore p = -5, q = 8}$

(d.) $\tan\theta = \sqrt{2} = \frac{\sqrt{2}}{1} = \frac{O}{A}$



$\sqrt{(\sqrt{2})^2 + 1^2} = \sqrt{3}$

$\therefore \sin\theta = \frac{\sqrt{2}}{\sqrt{3}}, \cos\theta = \frac{1}{\sqrt{3}}$

$\alpha = 1 + i\sqrt{2} = \sqrt{3}\left(\frac{1}{\sqrt{3}} + i\frac{\sqrt{2}}{\sqrt{3}}\right) = 3^{1/2}(\cos\theta + i\sin\theta)$

$\beta = 1 - i\sqrt{2} = \sqrt{3}\left(\frac{1}{\sqrt{3}} - i\frac{\sqrt{2}}{\sqrt{3}}\right) = 3^{1/2}(\cos\theta - i\sin\theta)$

$$\alpha^n = 3^{\frac{n}{2}}(\cos n\theta + i\sin n\theta)$$

$$\beta^n = 3^{\frac{n}{2}}(\cos n\theta - i\sin n\theta)$$

$$\alpha^n + \beta^n = 3^{\frac{n}{2}}(\cos n\theta + \cancel{i\sin n\theta}) + 3^{\frac{n}{2}}(\cos n\theta - \cancel{i\sin n\theta})$$

$$= 2\left(3^{\frac{n}{2}}\right)(\cos n\theta)$$

$$\gamma^n = \left(\frac{1}{2}\right)^n = \frac{1}{2^n}$$

$$\boxed{\therefore \alpha^n + \beta^n + \gamma^n = 2\left(3^{\frac{n}{2}}\right)(\cos n\theta) + \frac{1}{2^n}}$$

10 A particle of mass 0.5 kg is initially at point O. It moves from rest along the x-axis under the influence of two forces F_1 N and F_2 N which act parallel to the x-axis. At time t seconds the velocity of the particle is $v\,\mathrm{m\,s^{-1}}$.
F_1 is acting in the direction of motion of the particle and F_2 is resisting motion.

In an initial model

- F_1 is proportional to t with constant of proportionality $\lambda > 0$,
- F_2 is proportional to v with constant of proportionality $\mu > 0$.

(a) Show that the motion of the particle can be modelled by the following differential equation.

$$\frac{1}{2}\frac{\mathrm{d}v}{\mathrm{d}t} = \lambda t - \mu v$$ **[2]**

(b) Solve the differential equation in part **(a)**, giving the particular solution for v in terms of t, λ and μ. **[7]**

You are now given that $\lambda = 2$ and $\mu = 1$.

(c) Find a formula for an approximation for v in terms of t when t is large. **[2]**

In a refined model

- F_1 is constant, acting in the direction of motion with magnitude 2 N,
- F_2 is as before with $\mu = 1$.

(d) Write down a differential equation for the refined model. **[1]**

(e) Without solving the differential equation in part **(d)**, write down what will happen to the velocity in the long term according to this refined model. **[1]**

(a.) $F = ma$

$F = \lambda t - \mu v$

$$\boxed{\therefore \frac{1}{2}\frac{dv}{dt} = \lambda t - \mu v}$$

(b.) $\frac{dv}{dt} = 2\lambda t - 2\mu v$

$\frac{dv}{dt} + 2\mu v = 2\lambda t$

$IF = e^{\int p(x)\,dx} = e^{\int 2\mu\,dt} = e^{2\mu t}$

Multiply by IF:

$$e^{2\mu t}\frac{dv}{dt} + e^{2\mu t}2\mu v = 2\lambda t e^{2\mu t}$$

$$\frac{d}{dt}\left(e^{2\mu t}v\right) = 2\lambda t e^{2\mu t}$$

$$e^{2\mu t}v = \int 2\lambda t e^{2\mu t}\,dt$$

$$\cancel{e^{2\mu t}}v = 2\lambda\left(\frac{1}{2\mu}t\cancel{e^{2\mu t}} - \frac{1}{4\mu^2}\cancel{e^{2\mu t}}\right) + c$$

$\div e^{2\mu t}$ $\qquad$ $\div e^{2\mu t}$

Given $t=0, v=0 \Rightarrow 0 = 2\lambda\left(0 - \frac{1}{4\mu^2}\right) + c$

$$\therefore c = \frac{2\lambda}{4\mu^2} = \frac{\lambda}{2\mu^2}$$

$$v = 2\lambda\left(\frac{1}{2\mu}t - \frac{1}{4\mu^2}\right) + \left(\frac{\lambda}{2\mu^2} \div e^{2\mu t}\right)$$

$$v = \frac{\cancel{2}\lambda t}{\cancel{2}\mu} - \frac{2\lambda}{4\mu^2} + \frac{\lambda}{2\mu^2}e^{-2\mu t}$$

$$\boxed{\therefore v = \frac{\lambda t}{\mu} - \frac{\lambda}{2\mu^2} + \frac{\lambda e^{-2\mu t}}{2\mu^2}}$$

(c) $\lambda = 2, \mu = 1 \Rightarrow v = \frac{2t}{1} - \frac{2}{2(1)^2} + \frac{2e^{-2(1)t}}{2(1)^2} = 2t - 1 + e^{-2t}$

$\boxed{\text{When } t \text{ is large, } e^{-2t} \text{ is very small, } \therefore v \approx 2t-1.}$

(d.) $\frac{1}{2}\frac{dv}{dt} = \lambda t - \mu v$

$\lambda = 2,\ \mu = 1 \Rightarrow \boxed{\frac{1}{2}\frac{dv}{dt} = 2t - v}$

(e.) As v approaches 2, $\frac{dv}{dt} \to 0$, ie. v approches a constant value.

11 A curve has cartesian equation $x^3 + y^3 = 2xy$.

C is the portion of the curve for which $x \geqslant 0$ and $y \geqslant 0$. The equation of C in polar form is given by $r = \mathrm{f}(\theta)$ for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(a) Find $\mathrm{f}(\theta)$. **[2]**

(b) Find an expression for $\mathrm{f}\left(\frac{1}{2}\pi - \theta\right)$, giving your answer in terms of $\sin\theta$ and $\cos\theta$. **[2]**

(c) Hence find the line of symmetry of C. **[1]**

(d) Find the value of r when $\theta = \frac{1}{4}\pi$. **[1]**

(e) By finding values of θ when $r = 0$, show that C has a loop. **[2]**

(a.) $x^3 + y^3 = 2xy$

$x = r\cos\theta$, $y = r\sin\theta \Rightarrow (r\cos\theta)^3 + (r\sin\theta)^3 = 2(r\cos\theta)(r\sin\theta)$

$r^{\cancel{3}}(\cos^3\theta + \sin^3\theta) = 2\cancel{r^2}\cos\theta\sin\theta$

$$\boxed{\therefore r = \frac{2\cos\theta\sin\theta}{\cos^3\theta + \sin^3\theta}}$$

(b.) $$f\left(\frac{\pi}{2} - \theta\right) = \frac{2\cos\left(\frac{\pi}{2} - \theta\right)\sin\left(\frac{\pi}{2} - \theta\right)}{\cos^3\left(\frac{\pi}{2} - \theta\right) + \sin^3\left(\frac{\pi}{2} - \theta\right)}$$

$$\boxed{\therefore f\left(\frac{\pi}{2} - \theta\right) = \frac{2\sin\theta\cos\theta}{\sin^3\theta + \cos^3\theta}}$$

$\sin(90° - \theta) = \cos\theta$

$\cos(90° - \theta) = \sin\theta$

$\frac{\pi}{2}$ rads $= 90°$

(c.) $\frac{\pi}{2} \div 2 = \frac{\pi}{4}$

$\boxed{\therefore \text{Line of symmetry: } \theta = \frac{\pi}{4} \text{ or } x = y.}$

(d.) $\theta = \frac{\pi}{4} \Rightarrow r = \dfrac{2\cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right)}{\cos^3\left(\frac{\pi}{4}\right) + \sin^3\left(\frac{\pi}{4}\right)} = \dfrac{2\left(\frac{\sqrt{2}}{2}\right)^2}{2\left(\frac{\sqrt{2}}{2}\right)^3} = \sqrt{2}$

$\boxed{\therefore r = \sqrt{2}}$

(e.) $r = 0$ when $\theta = 0$.

$r = 0$ also when $\theta = \frac{\pi}{2}$.

In range $0 < \theta < \frac{\pi}{2}$, $r > 0$ and is continuous.

$\therefore$ There is a loop.

12 Show that $\displaystyle\int_0^{\frac{1}{\sqrt{3}}} \frac{4}{1-x^4}\,dx = \ln(a+\sqrt{b}) + \frac{\pi}{c}$ where a, b and c are integers to be determined. **[6]**

$$\frac{4}{1-x^4} \equiv \frac{A}{1-x} + \frac{B}{1+x} + \frac{Cx+D}{1+x^2}$$

$$A(1+x)(1+x^2) + B(1-x)(1+x^2) + (Cx+D)\overbrace{(1-x)(1+x)}^{(-x^2+1)} \equiv 4$$

$$A(x^3+x^2+x+1) + B(-x^3+x^2-x+1) + (-Cx^3 - Dx^2 + Cx + D) \equiv 4$$

$x^3: \; A - B - C = 0$ (1)

$x^2: \; A + B - D = 0$ (2) $\rightarrow D = A + B$

$x: \; A - B - C = 0$ (1)

$1: \; A + B + D = 4$ (3) $\rightarrow A + B + (A+B) = 4$

$2(A+B) = 4$

$\therefore A + B = 2 = D$

$x = 1: \; 4A + 0B + 0C + 0D \equiv 4 \Rightarrow \therefore A = 1$

$x = -1: \; 0A + 4B + 0C + 0D \equiv 4 \Rightarrow \therefore B = 1$

$\therefore C = A - B = 1 - 1 = 0$

$\int \frac{1}{1+x}dx = \ln|1+x| + c$

$\int \frac{1}{1+x^2}dx = \tan^{-1}x + c$

$$I = \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{1-x} + \frac{1}{1+x} + \frac{2}{1+x^2}\,dx$$

$$= \left[-\ln|1-x| + \ln|1+x| + 2\tan^{-1}x\right]_0^{\frac{1}{\sqrt{3}}}$$

$$= \left[\ln\left|\frac{1+x}{1-x}\right| + 2\tan^{-1}x\right]_0^{\frac{1}{\sqrt{3}}}$$

$$= \left[\ln|2+\sqrt{3}| + 2\left(\frac{\pi}{6}\right)\right] - \left[\ln|1| + 2(0)\right]$$

$$= \ln|2+\sqrt{3}| + \frac{\pi}{3}$$

$$\boxed{\therefore a = 2,\; b = 3,\; c = 3}$$