OCR
Oxford Cambridge and RSA

Accredited

A Level Further Mathematics A
Y543 Mechanics
Sample Question Paper

Date – Morning/Afternoon
Time allowed: 1 hour 30 minutes

MODEL ANSWERS

OCR supplied materials:
- Printed Answer Booklet
- Formulae A Level Further Mathematics A

You must have:
- Printed Answer Booklet
- Formulae A Level Further Mathematics A
- Scientific or graphical calculator

INSTRUCTIONS

- Use black ink. HB pencil may be used for graphs and diagrams only.
- Complete the boxes provided on the Printed Answer Booklet with your name, centre number and candidate number.
- Answer **all** the questions.
- **Write your answer to each question in the space provided in the Printed Answer Booklet.**
- Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- Do **not** write in the bar codes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The acceleration due to gravity is denoted by $g\,\mathrm{m\,s^{-2}}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION

- The total number of marks for this paper is **75**.
- The marks for each question are shown in brackets **[ ]**.
- **You are reminded of the need for clear presentation in your answers.**
- The Printed Answer Booklet consists of **12** pages. The Question Paper consists of **8** pages.

1 A body, P, of mass 2 kg moves under the action of a single force $\mathbf{F}$ N. At time t s, the velocity of the body is $\mathbf{v}$ m s^{-1}, where

$$\mathbf{v} = (t^2 - 3)\mathbf{i} + \frac{5}{2t+1}\mathbf{j} \text{ for } t \geq 2.$$

(i) Obtain $\mathbf{F}$ in terms of t. [3]

(ii) Calculate the rate at which the force $\mathbf{F}$ is working at $t = 4$. [3]

(iii) By considering the change in kinetic energy of P, calculate the work done by the force $\mathbf{F}$ during the time interval $2 \leq t \leq 4$. [3]

i. $\underline{a} = \frac{dv}{dt}$ $\underline{V} = \begin{pmatrix} t^3 - 3 \\ \frac{5}{2t+1} \end{pmatrix}$ $\therefore$ $\underline{a} = \begin{pmatrix} 2t \\ \frac{-10}{(2t+1)^2} \end{pmatrix}$

$\underline{F} = m\underline{a} \rightarrow \underline{F} = 2\begin{pmatrix} 2t \\ \frac{-10}{(2t+1)^2} \end{pmatrix} = \begin{pmatrix} 4t \\ \frac{-20}{(2t+1)^2} \end{pmatrix}$

$= 4t\underline{i} - \frac{20}{(2t+1)^2}\underline{j}$

ii. $p = \underline{F} \cdot \underline{V}$

$\underline{F} = \begin{pmatrix} 4t \\ \frac{-20}{(2t+1)^2} \end{pmatrix}$ $\underline{V} = \begin{pmatrix} t^3 \\ \frac{5}{2t+1} \end{pmatrix}$

@$t=4$: $\underline{F} = \begin{pmatrix} 16 \\ \frac{-20}{81} \end{pmatrix}$ $\underline{V} = \begin{pmatrix} 13 \\ \frac{5}{9} \end{pmatrix}$ $\therefore$ $\underline{F} \cdot \underline{v} = \begin{pmatrix} 16 \\ \frac{-20}{81} \end{pmatrix} \cdot \begin{pmatrix} 13 \\ \frac{5}{9} \end{pmatrix} = 208\text{W (3sf)}$

iii. $KE = \frac{1}{2}mv^2$ & ΔKE = work done by F

$\Delta KE = \frac{1}{2}m(V_4^2 - V_2^2)$

$V(t=4) = \begin{pmatrix} 13 \\ \frac{5}{9} \end{pmatrix}$ $\therefore$ $V^2(t=4) = 13^2 + \left(\frac{5}{9}\right)^2 = 169.3$

$$V(t=2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \therefore \quad V^2(t=2) = 2$$

$$\therefore \quad \Delta KE = \frac{1}{2} \times 2(169.3 - 2) = 167\,J \quad (3\text{ s.f.})$$

2 As part of a training exercise an army recruit of mass 75 kg falls a vertical distance of 5 m before landing on a mat of thickness 1.2 m. The army recruit sinks a distance of x m into the mat before instantaneously coming to rest. The mat can be modelled as a spring of natural length 1.2 m and modulus of elasticity 10 800 N and the army recruit can be modelled as a particle falling vertically with an initial speed of 2 m s^{-1}.

(i) Show that x satisfies the equation $300x^2 - 49x - 255 = 0$. **[5]**

(ii) Calculate the value of x. **[2]**

(iii) Ignoring the possible effect of air resistance, make
- one comment on the assumptions made and,
- suggest a possible refinement to the model. **[2]**

i. Conservation of energy: $GPE_0 + KE_0 = GPE_1 + EPE_1$

$$75g \times 6.2 + \frac{1}{2} \times 2^2 \times 75 = 75g(1.2 - x) + \frac{10800}{2 \times 1.2} x^2$$

$$4707 = 882 - 735x + 4500x^2$$

$$300x^2 - 49x - 255 = 0 \quad \text{(divide through by 15)}$$

ii. $x > 0$

$\therefore x = 1.01$, $x \neq -0.844$ $\because$ this isn't a valid value

iii. $g = 9.8$ isn't precise enough, use $g = 9.81$

(e.g.) the recruit isn't a particle, model as a rigid body

3 A body, Q, of mass 2 kg moves in a straight line under the action of a single force which acts in the direction of motion of Q. Initially the speed of Q is $5\,\text{m s}^{-1}$. At time t s, the magnitude F N of the force is given by

$$F = t^2 + 3e^t, \quad 0 \le t \le 4.$$

(i) Calculate the impulse of the force over the time interval. **[3]**

(ii) Hence find the speed of Q when $t = 4$. **[2]**

i. $I = \int_0^4 F\,dt = \left[\frac{t^3}{3} + 3e^t\right]_0^4$

$= \frac{55}{3} + 3e^4$

$= 182\text{ Ns } (3\text{s.f.})$

ii. $\underline{I} = \Delta\text{momentum}$

$\frac{55}{3} + 3e^4 = 2(V_0 - 5)$

$V_0 = 96.1\text{ ms}^{-1}$

4 A light inextensible taut rope, of length 4 m, is attached at one end A to the centre of the horizontal ceiling of a gym. The other end of the rope B is being held by a child of mass 35 kg. Initially the child is held at rest with the rope making an angle of 60° to the downward vertical and it may be assumed that the child can be modelled as a particle attached to the end of the rope. The child is released at a height 5 m above the horizontal ground.

(i) Show that the speed, $v\,\text{m s}^{-1}$, of the child when the rope makes an angle θ with the downward vertical is given by $v^2 = 4g(2\cos\theta - 1)$. **[5]**

(ii) At the instant when $\theta = 0^\circ$, the child lets go of the rope and moves freely under the influence of gravity only. Determine the speed and direction of the child at the moment that the child reaches the ground. **[5]**

(iii) The child returns to the initial position and is released again from rest. Find the value of θ when the tension in the rope is three times greater than the tension in the rope at the instant the child is released. **[5]**

i. $mgh_1 = mgh_2 + \frac{1}{2}mv^2$

$2gh_1 = 2gh_2 + v^2$

$8g(1-\cos 60) = 8g(1-\cos\theta) + v^2$

$v^2 = 4g(2\cos\theta - 1)$

ii. @ $\theta = 0$, $V_H = 2\sqrt{g}$

$V_v^2 = 2g(3)$

$|V| = \sqrt{V_v^2 + V_h^2} = \sqrt{10g} = 9.90\,\text{ms}^{-1}$

$\tan\theta = \frac{V_v}{V_H} \Rightarrow \theta = \arctan\sqrt{\frac{6g}{4g}} = 50.8^\circ$ below the horizontal

iii. $T_1 - 35g\cos 60 = 35g(2\cos 60 - 1)$

$T_1 = 17.5g$

$T_2 - 35g\cos\theta = 35g(2\cos\theta - 1)$

$T_2 = 35g(3\cos\theta - 1)$

$$3T_1 = T_2$$

$$\rightarrow 3 \times 17.5g = 35g(3\cos\theta - 1)$$

$$\frac{3}{2} = 3\cos\theta - 1$$ (divide through by 35g)

$$\cos\theta = \frac{5}{6}$$

$$\theta = 33.6^\circ$$

5 A particle P of mass m kg is projected vertically upwards through a liquid. Student A measures P's initial speed as (8.5 ± 0.25) m s^{-1} and they also record the time for P to attain its greatest height above the initial point of projection as over 3 seconds.
In an attempt to model the motion of P student B determines that t seconds after projection the only forces acting on P are its weight and the resistance from the liquid. Student B models the resistance from the liquid to be of magnitude $mv^2 - 6mv$, where v is the speed of the particle.

(i) (a) Show that $\frac{dt}{dv} = -\frac{1}{(v-3)^2 + 0.8}$. **[3]**

(b) Determine whether student B's model is consistent with the time recorded by A for P to attain its greatest height. **[5]**

i. a) $ma = m\frac{dv}{dt}$ $\quad F = m\frac{dv}{dt}$

$$-mg - (-6mv + mv^2) = F$$

$$\frac{dv}{dt} = -v^2 + 6v - 9.8$$
$$= -(v-3)^2 - 0.8$$

$$\therefore \frac{dt}{dv} = \frac{-1}{(v-3)^2 + 0.8}$$

b) $t = \int \frac{dv}{(v-3)^2 + 0.8}$

$$t_1 = -\int_{8.25}^{0} \frac{dv}{(v-3)^2 + 0.8}$$
$$= 2.9998$$

$$t_2 = -\int_{8.75}^{0} \frac{dv}{(v-3)^2 + 0.8}$$
$$= 3.0159$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$2.9998 < t < 3.0159$, so B's Model for majority of speeds in ± 0.25 margin does give $t > 3$ $\therefore$ it is likely but not certain that model is consistent with A's time

After attaining its greatest height P now falls through the liquid. Student C claims that the time taken for P to achieve a speed of 1 m s^{-1} when falling through the liquid is given by

$$-\int_0^1 \frac{1}{(v-3)^2+0.8}\,\mathrm{d}v.$$

(ii) Explain why student C's claim is incorrect and write down the integral which would give the correct time for P to achieve a speed of 1 m s^{-1} when falling through the liquid. **[3]**

ii. C's claim is incorrect as they have assumed the differential equation for motion is the same in each direction

D.E. for falling is $mg-(-6mv+mv^2)=m\frac{dv}{dt}$

$-\int_0^1 \frac{1}{(v-3)^2-18.8}$ is correct integral.

6 Two uniform smooth spheres A and B of equal radius are moving on a smooth horizontal surface when they collide. A has mass 2.5 kg and B has mass 3 kg. Immediately before the collision A and B each has speed $u\,\mathrm{m\,s^{-1}}$ and each moves in a direction at an angle θ to their line of centres, as indicated in Fig. 1. Immediately after the collision A has speed $v_1\,\mathrm{m\,s^{-1}}$ and moves in a direction at an angle α to the line of centres, and B has speed $v_2\,\mathrm{m\,s^{-1}}$ and moves in a direction at an angle β to the line of centres as indicated in Fig. 2. The coefficient of restitution between A and B is e.

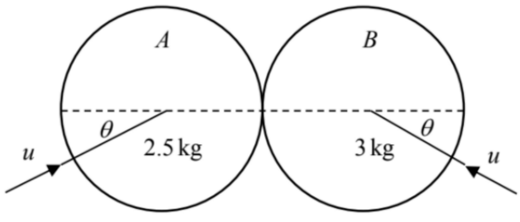

Fig. 1

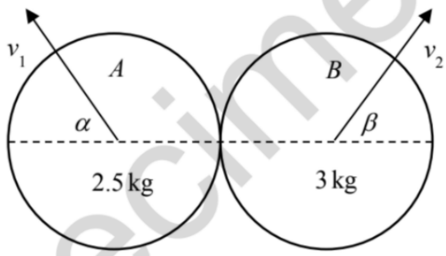

Fig. 2

(i) Show that $\tan\beta = \dfrac{11\tan\theta}{10e-1}$. **[8]**

(ii) Given that after impact sphere A moves at an angle of 50° to the line of centres and B moves perpendicular to the line of centres, find θ. **[4]**

i. use conservation of linear momentum & restitution equation

$u\sin\theta = V_1\sin\alpha$ & $u\sin\theta = V_2\sin\beta$

$$\underbrace{2.5u\cos\theta - 3u\cos\theta}_{-0.5u\cos\theta} = -2.5V_1\cos\alpha + 3V_2\cos\beta \quad ①$$

$$V_1\cos\alpha + V_2\cos\beta = e(u\cos\theta + u\cos\theta) \quad ②$$

$$① - 2.5\times② : \; 5.5V_2\cos\beta = u\cos\theta\left(5e - \tfrac{1}{2}\right)$$

$$\cos\beta = \frac{u\cos\theta}{11V_2}(10e-1)$$

$$\tan\beta = \frac{\sin\beta}{\cos\beta} = \frac{\frac{u\sin\theta}{V_2}}{\frac{u\cos\theta(10e-1)}{11V_2}}$$

$$= \frac{11\tan\theta}{10e-1}$$

ii. perpendicular

$$\Rightarrow 10e-1=0 \quad \therefore e=\frac{1}{10}$$

$$\tan\alpha = \frac{11\tan\theta}{1+12e}$$

$$\tan 50 = \frac{11\tan\theta}{1+\frac{12}{10}}$$

$$\therefore \theta = 13.4$$

7

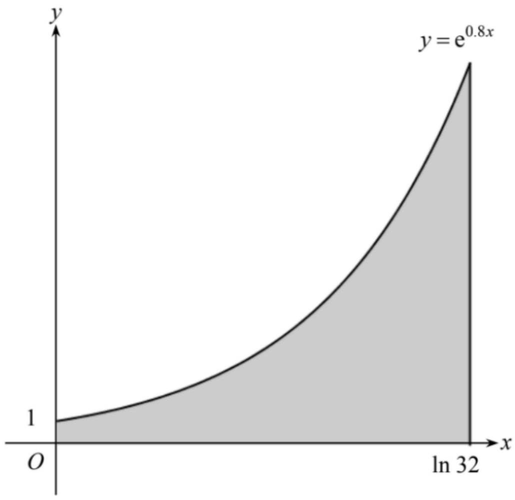

Fig. 3

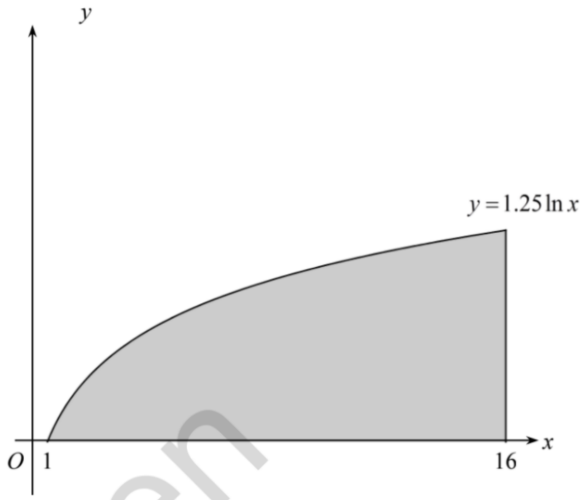

Fig. 4

The region bounded by the x-axis, the y-axis, the line $x = \ln 32$ and the curve $y = e^{0.8x}$ for $0 \le x \le \ln 32$, is occupied by a uniform lamina (see Fig. 3).

(i) Show that the x-coordinate of the centre of mass of the lamina is given by $\frac{16}{3}\ln 2 - \frac{5}{4}$. **[7]**

(ii) Calculate the y-coordinate of the centre of mass of the lamina. **[3]**

(iii) The region bounded by the x-axis, the line $x = 16$ and the curve $y = 1.25 \ln x$ for $1 \le x \le 16$, is occupied by a second uniform lamina (see Fig. 4). By using your answer to part **(i)** find, to 3 significant figures, the x-coordinate of the centre of mass of this second lamina. **[4]**

END OF QUESTION PAPER

i. $$A = \int_0^{\ln 32} e^{0.8x}\,dx = \frac{75}{4}$$

$$A\bar{x} = \int_0^{\ln 32} xe^{0.8x}\,dx = \left[1.25xe^{0.8x}\right]_0^{\ln 32} - 1.25\int_0^{\ln 32} e^{0.8x}\,dx$$

by parts

$$= \left[1.25xe^{0.8x} - 1.5625e^{0.8x}\right]_0^{\ln 32}$$

$$= (20\ln 32 - 25) - (0 - 1.5625)$$

$$\Rightarrow \bar{x} = \frac{16}{3}\ln 2 - \frac{5}{4}$$

ii. $$A\bar{y} = \int_0^{\ln 32} (e^{0.8x})^2 \, dx = \frac{1275}{16}$$

$$\frac{A\bar{y}}{A} = \frac{\frac{1275}{16}}{\frac{75}{4}} = \frac{17}{4}$$

iii. $$A = \int_1^{16} 1.25\ln x = 36.702$$

$$A\bar{x} = \int_1^{16} 1.25x\ln x = 363.9267$$

$$\bar{x} = \frac{A\bar{x}}{A} = \frac{363.9267}{36.702}$$

$$\bar{x} = 9.92$$