Accredited

A Level Further Mathematics A
Y545 Additional Pure Mathematics
Sample Question Paper

Date – Morning/Afternoon
Time allowed: 1 hour 30 minutes

OCR supplied materials:
Printed Answer Booklet
Formulae A Level Further Mathematics A

You must have:
• Printed Answer Booklet
• Formulae A Level Further Mathematics A
• Scientific or graphical calculator

INSTRUCTIONS

• Use black ink. HB pencil may be used for graphs and diagrams only.
• Complete the boxes provided on the Printed Answer Booklet with your name, centre number and candidate number.
• Answer **all** the questions.
• **Write your answer to each question in the space provided in the Printed Answer Booklet.**
• Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
• Do **not** write in the bar codes.
• You are permitted to use a scientific or graphical calculator in this paper.
• Final answers should be given to a degree of accuracy appropriate to the context.
• The acceleration due to gravity is denoted by $g\,\text{ms}^{-2}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION

• The total number of marks for this paper is **75**.
• The marks for each question are shown in brackets **[ ]**.
• **You are reminded of the need for clear presentation in your answers.**
• The Printed Answer Booklet consists of **12** pages. The Question Paper consists of **4** pages.

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Answer **all** the questions.

1   A curve is given by $x = t^2 - 2\ln t$, $y = 4t$ for $t > 0$. When the arc of the curve between the points where $t = 1$ and $t = 4$ is rotated through 2π radians about the x-axis, a surface of revolution is formed with surface area A.
Given that $A = k\pi$, where k is an integer,
    - write down an integral which gives A and
    - find the value of k.   **[4]**

2   Find the volume of tetrahedron OABC, where O is the origin, A = (2, 3, 1), B = (–4, 2, 5) and C = (1, 4, 4).
                                                                                                                 [3]

3   Given $z = x\sin y + y\cos x$, show that $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} + z = 0$.   **[5]**

4   **(i)** Solve the recurrence relation $u_{n+2} = 4u_{n+1} - 4u_n$ for $n \geq 0$, given that $u_0 = 1$ and $u_1 = 1$.   **[4]**

   (ii) Show that each term of the sequence $\{u_n\}$ is an integer.   **[2]**

5   **In this question you must show detailed reasoning.**
   It is given that $I_n = \displaystyle\int_0^\pi \sin^n \theta \, d\theta$ for $n \geq 0$.

   (i) Prove that $I_n = \dfrac{n-1}{n} I_{n-2}$ for $n \geq 2$.   **[5]**

   (ii) (a) Evaluate I_1.   **[2]**

   (b) Use the reduction formula to determine the exact value of $\displaystyle\int_0^\pi \cos^2 \theta \, \sin^5 \theta \, d\theta$.   **[2]**

6   A surface S has equation $z = f(x, y)$, where $f(x, y) = 2x^2 - y^2 + 3xy + 17y$. It is given that S has a single stationary point, P.

   (i) (a) Determine the coordinates of P.   **[5]**

   (b) Determine the nature of P.   **[3]**

   (ii) Find the equation of the tangent plane to S at the point $Q(1, 2, 38)$.   **[2]**

7 In order to rescue them from extinction, a particular species of ground-nesting birds is introduced into a nature reserve. The number of breeding pairs of these birds in the nature reserve, t years after their introduction, is an integer denoted by N_t. The initial number of breeding pairs is given by N_0.

An initial discrete population model is proposed for N_t.

$$\text{Model I: } N_{t+1} = \tfrac{6}{5} N_t \left(1 - \tfrac{1}{900} N_t\right)$$

 (i) (a) For Model I, show that the steady state values of the number of breeding pairs are 0 and 150. **[3]**

  (b) Show that $N_{t+1} - N_t < 150 - N_t$ when N_t lies between 0 and 150. **[3]**

  (c) Hence find the long-term behaviour of the number of breeding pairs of this species of birds in the nature reserve predicted by Model I when $N_0 \in (0, 150)$. **[2]**

An alternative discrete population model is proposed for N_t.

$$\text{Model II: } N_{t+1} = \text{INT}\left(\tfrac{6}{5} N_t \left(1 - \tfrac{1}{900} N_t\right)\right)$$

 (ii) (a) Given that $N_0 = 8$, find the value of N_4 for each of the two models. **[2]**

  (b) Which of the two models gives values for N_t with the more appropriate level of precision? **[1]**

8 The set X consists of all 2×2 matrices of the form $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$, where x and y are real numbers which are

not **both** zero.

   (i) (a) The matrices $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ and $\begin{pmatrix} c & -d \\ d & c \end{pmatrix}$ are both elements of X.

      Show that $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} c & -d \\ d & c \end{pmatrix} = \begin{pmatrix} p & -q \\ q & p \end{pmatrix}$ for some real numbers p and q to be found in terms of

      a, b, c and d. **[2]**

     (b) Prove by contradiction that p and q are not **both** zero. **[5]**

  (ii) Prove that X, under matrix multiplication, forms a group G.
     [You may use the result that matrix multiplication is associative.] **[4]**

 (iii) Determine a subgroup of G of order 17. **[2]**

9   **(i) (a)** Prove that $p \equiv \pm 1 \,(\mathrm{mod}\,6)$ for all primes $p > 3$. **[2]**

     (b) Hence or otherwise prove that $p^2 - 1 \equiv 0 \,(\mathrm{mod}\,24)$ for all primes $p > 3$. **[3]**

  (ii) Given that p is an odd prime, determine the residue of 2^{p^2-1} modulo p. **[4]**

 (iii) Let p and q be distinct primes greater than 3. Prove that $p^{q-1} + q^{p-1} \equiv 1 \,(\mathrm{mod}\,pq)$. **[5]**

END OF QUESTION PAPER