GCE

Further Mathematics A

Y545/01: Additional Pure Mathematics

Advanced GCE

Mark Scheme for Autumn 2021

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This mark scheme is published as an aid to teachers and students, to indicate the requirements of the examination. It shows the basis on which marks were awarded by examiners. It does not indicate the details of the discussions which took place at an examiners' meeting before marking commenced.

All examiners are instructed that alternative correct answers and unexpected approaches in candidates' scripts must be given marks that fairly reflect the relevant knowledge and skills demonstrated.

Mark schemes should be read in conjunction with the published question papers and the report on the examination.

1. Annotations and abbreviations

Annotation in RM assessor	Meaning
✓and ✖	
BOD	Benefit of doubt
FT	Follow through
ISW	Ignore subsequent working
M0, M1	Method mark awarded 0, 1
A0, A1	Accuracy mark awarded 0, 1
B0, B1	Independent mark awarded 0, 1
SC	Special case
^	Omission sign
MR	Misread
BP	Blank Page
Seen	
Highlighting	
Other abbreviations in mark scheme	**Meaning**
dep*	Mark dependent on a previous mark, indicated by *. The * may be omitted if only one previous M mark
cao	Correct answer only
oe	Or equivalent
rot	Rounded or truncated
soi	Seen or implied
www	Without wrong working
AG	Answer given
awrt	Anything which rounds to
BC	By Calculator
DR	This question included the instruction: In this question you must show detailed reasoning.

2. Subject-specific Marking Instructions for A Level Mathematics A

a Annotations must be used during your marking. For a response awarded zero (or full) marks a single appropriate annotation (cross, tick, M0 or ^) is sufficient, but not required.

For responses that are not awarded either 0 or full marks, you must make it clear how you have arrived at the mark you have awarded and all responses must have enough annotation for a reviewer to decide if the mark awarded is correct without having to mark it independently.

It is vital that you annotate standardisation scripts fully to show how the marks have been awarded.

Award NR (No Response)

- if there is nothing written at all in the answer space and no attempt elsewhere in the script
- OR if there is a comment which does not in any way relate to the question (e.g. 'can't do', 'don't know')
- OR if there is a mark (e.g. a dash, a question mark, a picture) which isn't an attempt at the question.

Note: Award 0 marks only for an attempt that earns no credit (including copying out the question).

If a candidate uses the answer space for one question to answer another, for example using the space for 8(b) to answer 8(a), then give benefit of doubt unless it is ambiguous for which part it is intended.

b An element of professional judgement is required in the marking of any written paper. Remember that the mark scheme is designed to assist in marking incorrect solutions. Correct solutions leading to correct answers are awarded full marks but work must not always be judged on the answer alone, and answers that are given in the question, especially, must be validly obtained; key steps in the working must always be looked at and anything unfamiliar must be investigated thoroughly. Correct but unfamiliar or unexpected methods are often signalled by a correct result following an apparently incorrect method. Such work must be carefully assessed. When a candidate adopts a method which does not correspond to the mark scheme, escalate the question to your Team Leader who will decide on a course of action with the Principal Examiner.
If you are in any doubt whatsoever you should contact your Team Leader.

c The following types of marks are available.

M

A suitable method has been selected and *applied* in a manner which shows that the method is essentially understood. Method marks are not usually lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. In some cases the nature of the errors allowed for the award of an M mark may be specified.
A method mark may usually be implied by a correct answer unless the question includes the DR statement, the command words "Determine" or "Show that", or some other indication that the method must be given explicitly.

A

Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated Method mark is earned (or implied). Therefore M0 A1 cannot ever be awarded.

B

Mark for a correct result or statement independent of Method marks.

Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored. Sometimes this is reinforced in the mark scheme by the abbreviation isw. However, this would not apply to a case where a candidate passes through the correct answer as part of a wrong argument.

d When a part of a question has two or more 'method' steps, the M marks are in principle independent unless the scheme specifically says otherwise; and similarly where there are several B marks allocated. (The notation 'dep*' is used to indicate that a particular mark is dependent on an earlier, asterisked, mark in the scheme.) Of course, in practice it may happen that when a candidate has once gone wrong in a part of a question, the work from there on is worthless so that no more marks can sensibly be given. On the other hand, when two or more steps are successfully run together by the candidate, the earlier marks are implied and full credit must be given.

e The abbreviation FT implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A and B marks are given for correct work only – differences in notation are of course permitted. A (accuracy) marks are not given for answers obtained from incorrect working. When A or B marks are awarded for work at an intermediate stage of a solution, there may be various alternatives that are equally acceptable. In such cases, what is acceptable will be detailed in the mark scheme. If this is not the case please, escalate the question to your Team Leader who will decide on a course of action with the Principal Examiner.
Sometimes the answer to one part of a question is used in a later part of the same question. In this case, A marks will often be 'follow through'. In such cases you must ensure that you refer back to the answer of the previous part question even if this is not shown within the image zone. You may find it easier to mark follow through questions candidate-by-candidate rather than question-by-question.

f We are usually quite flexible about the accuracy to which the final answer is expressed; over-specification is usually only penalised where the scheme explicitly says so.

- When a value **is given** in the paper only accept an answer correct to at least as many significant figures as the given value.
- When a value **is not given** in the paper accept any answer that agrees with the correct value to **3 s.f.** unless a different level of accuracy has been asked for in the question, or the mark scheme specifies an acceptable range.
  NB for Specification B (MEI) the rubric is not specific about the level of accuracy required, so this statement reads "2 s.f".

Follow through should be used so that only one mark in any question is lost for each distinct accuracy error.
Candidates using a value of 9.80, 9.81 or 10 for *g* should usually be penalised for any final accuracy marks which do not agree to the value found with 9.8 which is given in the rubric.

g Rules for replaced work and multiple attempts:

- If one attempt is clearly indicated as the one to mark, or only one is left uncrossed out, then mark that attempt and ignore the others.
- If more than one attempt is left not crossed out, then mark the last attempt unless it only repeats part of the first attempt or is substantially less complete.
- if a candidate crosses out all of their attempts, the assessor should attempt to mark the crossed out answer(s) as above and award marks appropriately.

h For a genuine misreading (of numbers or symbols) which is such that the object and the difficulty of the question remain unaltered, mark according to the scheme but following through from the candidate's data. A penalty is then applied; 1 mark is generally appropriate, though this may differ for some units. This is achieved by withholding one A or B mark in the question. Marks designated as cao may be awarded as long as there are no other errors.
If a candidate corrects the misread in a later part, do not continue to follow through. Note that a miscopy of the candidate's own working is not a misread but an accuracy error.

i If a calculator is used, some answers may be obtained with little or no working visible. Allow full marks for correct answers, provided that there is nothing in the wording of the question specifying that analytical methods are required such as the bold "In this question you must show detailed reasoning", or the command words "Show" or "Determine". Where an answer is wrong but there is some evidence of method, allow appropriate method marks. Wrong answers with no supporting method score zero. If in doubt, consult your Team Leader.

j If in any case the scheme operates with considerable unfairness consult your Team Leader.

Question			Answer	Marks	AOs	Guidance
1			$41723 = 10\times16^3 + 2\times16^2 + 15\times16 + 11$ $= A2FB_{16}$	**M1** **A1** **A1** **[3]**	1.1 1.1 1.2	Writing in terms of powers of 16; correct
2	(a)		$\{e, a, a^2\}$ $\{e, b\}$ $\{e, ab\}$ $\{e, a^2b\}$	**B1** **B1** **B1** **B1** **[4]**	1.1 1.1 1.1 1.1	Ignore inclusion of $\{e\}$, G
	(b)		G is isomorphic to the symmetric group of three elements i.e. D_3 (the symmetries of the Δ) or S_3 (the permutations of 3 symbols)	**B1** **[1]**	2.5	Accept "the non-cyclic group of order 6"
3	(a)		$\mathbf{p}\times\mathbf{q} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 1 & 5\\ 1 & -1 & 1\end{vmatrix} = \begin{pmatrix}6\\3\\-3\end{pmatrix}$ $= 3\begin{pmatrix}2\\1\\-1\end{pmatrix} \Rightarrow t = -1$	**M1** **A1** **[2]**	3.1a 2.2a	Possibly **BC** Correct vector product; t correct
	(b)		Vol. $OABC = \frac{1}{6}\lvert\mathbf{p}\times\mathbf{q}\,.\,\mathbf{r}\rvert = \frac{1}{6}\left\lvert\begin{pmatrix}6\\3\\-3\end{pmatrix}\bullet\begin{pmatrix}2\\1\\t\end{pmatrix}\right\rvert$ $= \frac{1}{6}\lvert 12 + 3 - 3t\rvert$ Solving $5 - t = 26$ and/or $t - 5 = 26$ $\Rightarrow t = -21$ or 31	**B1** **M1** **M1** **A1** **[4]**	1.1 1.1 3.1a 2.2a	A correct scalar triple product involving t Correct formula attempted Method may be implied by one correct t

Question	Answer	Marks	AOs	Guidance
4	$x = 3a + 1 = 11b + 5$	M1	1.1	Equating first two equations
	mod 11: $3a \equiv 4 \equiv 15 \Rightarrow a \equiv 5 \pmod{11}$ (since hcf(3, 11) = 1) $(a = 11c + 5)$	M1	2.1	Working mod 11
	$x = 33c + 16$	A1	3.1a	**or** via $b = 3c + 1$ (working mod 3)
	$2x = 66c + 32 = 17d + 5$ mod 17: $15c \equiv -27 \Rightarrow 5c \equiv -9 \equiv 8 \equiv 25$ $\Rightarrow c \equiv 5 \pmod{17}$ (since hcf(5, 17) = 1)	M1	2.1	Substituting into 3rd equation and working mod 17
	Then $c = 17e + 5 \Rightarrow x = 561e + 181$ **or** $x \equiv 181 \pmod{561}$	A1	2.2a	
	Stating at least one of either hcf(3,11) = 1 or hcf(5,17)=1	B1	2.4	
	Alternative Method 1			
	$2x \equiv 5 \equiv 22 \Rightarrow x \equiv 11 \pmod{17} \equiv -6 \pmod{17}$	M1		
	Then $x \equiv -6 \pmod{11}$	M1		
	$x \equiv -6 \pmod{187} \equiv 181$	A1		
	since hcf(11, 17) = 1	B1		
	Note that 181 is already $\equiv 1 \pmod 3$	M1		**OR** continue as above: $x = 3a + 1 = 187b + 181$
	Since there is a unique solution (mod $3 \times 11 \times 17 = 561$)			(as 3, 11, 17 are pairwise co-prime)
	the solution is $x \equiv 181 \pmod{561}$	A1		
	Alternative Method2 The *Chinese Remainder Theorem* (CRT)			
	There is a unique solution mod $M = 3 \times 11 \times 17 = 561$	B1		
	$2x \equiv 5 \equiv 22 \Rightarrow x \equiv 11 \pmod{17}$	M1		
	$a_1 = 1,\ M_1 = \frac{M}{3} = 187$, R_1 is the reciprocal of M_1 mod 3 = 1	M1		Since $187 \equiv 1 \pmod 3$ anyway For any one set of terms correct
	$a_2 = 5,\ M_2 = \frac{M}{11} = 51$, R_2 is the reciprocal of M_2 mod 11 = 8			Since $51 \equiv 7 \pmod{11}$ and $7 \times 8 = 56 \equiv 1 \pmod{11}$
	$a_3 = 11,\ M_3 = \frac{M}{17} = 33$, R_3 is the reciprocal of M_3 mod 17 = 16	A1		All three sets of terms correct. Since $33 \equiv -1 \equiv 16 \pmod{17}$
	Then $x \equiv a_1 M_1 R_1 + a_2 M_2 R_2 + a_3 M_3 R_3 \pmod{561}$	M1		CRT employed correctly (in principle)
	$= 1\times187\times1 + 5\times51\times8 + 11\times33\times16 = 8035 \equiv 181 \pmod{561}$	A1		**BC**
		[6]		

Question			Answer	Marks	AOs	Guidance
5	(a)		Differentiating $x^2+y^2+z^2=xyz-1$ partially w.r.t. either x or y	M1	1.1	Attempt with LHS correct
			x): $2x+2z\dfrac{\partial z}{\partial x}=xy\dfrac{\partial z}{\partial x}+yz$ ❶ y): $2y+2z\dfrac{\partial z}{\partial y}=xy\dfrac{\partial z}{\partial y}+xz$ ❷	A1 A1	1.1 1.1	First correct Correct or **FT** $x \leftrightarrow y$ (by symmetry)
			$x.❶+y.❷ \Rightarrow 2\left(x^2+y^2\right)+2z\left(x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}\right)=xy\left(x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}\right)+2xyz$	M1	3.1a	$\Rightarrow (2z-xy)\left(x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}\right)=2\left(xyz-x^2-y^2\right)$
			Now $x^2+y^2+z^2=xyz-1 \Rightarrow xyz-x^2-y^2=z^2+1$	M1	1.1	
			giving $(2z-xy)\left(x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}\right)=2\left(1+z^2\right)$	A1 [6]	2.2a	**AG** legitimately obtained
	(b)		If both $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ are zero …	M1	2.1	Considering conditions for stationary points
			… then LHS = 0 while RHS ≥ 2 (i.e. > 0) $(\Rightarrow\Leftarrow)$	A1 [2]	2.4	Contradiction justified
6	(a)		Closed … *YES* since the 'product' of 2 complex numbers is a third	B1 [1]	2.4	'Yes' with valid reason
	(b)		$[(a+\mathrm{i}b)\lozenge(c+\mathrm{i}d)]\lozenge(e+\mathrm{i}f)=[ac+\mathrm{i}(b+ad)]\lozenge(e+\mathrm{i}f)$ $=ace+\mathrm{i}(b+ad+acf)$	M1 A1	1.2 1.1	Attempt at first of $(pq)r$ or $p(qr)$, correct to here
			$(a+\mathrm{i}b)\lozenge[(c+\mathrm{i}d)\lozenge(e+\mathrm{i}f)]=(a+\mathrm{i}b)\lozenge[ce+\mathrm{i}(d+cf)]$ $=ace+\mathrm{i}(b+a[d+cf])=\ldots$	M1 A1 [4]	2.1 1.1	Attempt at the other, correct to here Must be convincingly shown equal to first answer
	(c)		$1+\mathrm{i}0$ is the identity	B1	2.2a	
			since $(1+\mathrm{i}0)\lozenge(a+\mathrm{i}b)=1\cdot a+\mathrm{i}(0+1b)=(a+\mathrm{i}b)\lozenge(1+\mathrm{i}0)$	B1 [2]	2.4	Demonstrated (condone one-side only)
	(d)		$(a+\mathrm{i}b)\lozenge(c+\mathrm{i}d)=1 \Rightarrow ac+\mathrm{i}(b+ad)=1$	M1	3.1a	Test for an inverse of a general element
			$\Rightarrow c=\dfrac{1}{a}$ and $d=\dfrac{-b}{a}$	A1	2.2a	Both components of inverse element found
			S is the set of complex numbers with non-zero real part (The 'exclusions' are those complex numbers with zero real part; i.e. the strictly imaginary numbers))	B1 [3]	3.2a	Explained

Question			Answer	Marks	AOs	Guidance
7	(a)		$I_n = \int_0^{\frac{1}{2}\pi} \cos^n x \, dx = \int_0^{\frac{1}{2}\pi} \cos^{n-1} x \cos x \, dx$	**M1**	1.1	Correct splitting and attempted use of 'parts'
			$= \left[\cos^{n-1} x \sin x\right] - \int_0^{\frac{1}{2}\pi} (n-1)\cos^{n-2} x.(-\sin x) \sin x \, dx$	**A1**	1.1	Correct unsimplified
			$= 0 + (n-1)\int_0^{\frac{1}{2}\pi} \cos^{n-2} x\left(1-\cos^2 x\right)dx = (n-1)\{I_{n-2} - I_n\}$	**M1**	3.1a	Setting up into I_k terms only
			$\Rightarrow n I_n = (n-1) I_{n-2}$	**A1** **[4]**	1.1	**AG** legitimately obtained
	(b)		$I_0 = \frac{1}{2}\pi$ evaluated and $I_n = \left(\frac{n-1}{n}\right) I_{n-2}$ used repeatedly	**M1**	1.1	Note: candidates may choose to evaluate $I_2 = \frac{1}{4}\pi$ directly as a starting-point
			gives $I_2 = \frac{1}{4}\pi$, $I_4 = \frac{3}{16}\pi$, $I_6 = \frac{5}{32}\pi$, $I_8 = \frac{35}{256}\pi$	**A1** **[2]**	1.1	
	(c)		$\int_0^{\frac{1}{2}\pi} \cos^6 x \sin^2 x \, dx = \{I_6 - I_8\}$	**M1**	3.1a	Converting into I_k terms and using the previous results from **(b)**
			$= \frac{5}{32}\pi - \frac{35}{256}\pi = \frac{5}{256}\pi$	**A1** **[2]**	2.2a	NB Missing factor of 2 and/or incorrect sign penalised only here at the end **FT** $I_6 - I_8 = \frac{1}{7} I_8 = \frac{1}{7}$ (answer fom **(b)**)

Question			Answer	Marks	AOs	Guidance
8	(a)		Aux. Eqn. is $m^2 - 5m + 4 = 0 \Rightarrow m = 1, 4$	**M1**	1.1	
			$\Rightarrow$ Gen. Soln. is $H_n = A + B \times 4^n$	**A1**	1.1	
			$H_0 = 3 \Rightarrow 3 = A + B$ and $H_1 = 7 \Rightarrow 7 = A + 4B$	**M1**	1.1	Use of initial terms
			$\Rightarrow A = \frac{5}{3}, \quad B = \frac{4}{3}$	**M1**	1.1	Solving simultaneous eqns. in A and B
			giving $H_n = \frac{1}{3}\left(5 + 4^{n+1}\right)$	**A1**	1.1	
				[5]		
	(b)	(i)	0, 1, 4, 5, 6, 9	**B1**	1.1	In any order
				[1]		
		(ii)	For $n = 0, 1, 2, \ldots$ $4^n \equiv 4, 6, 4, 6, \ldots \pmod{10}$	**M1**	3.1a	
			$\Rightarrow 5 + 4^n = 9, 11, 9, 11, \ldots \pmod{10}$	**A1**	1.1	
			$\equiv 9, 21, 9, 21, \ldots \pmod{10}$	**M1**	1.1	
			$\Rightarrow \frac{1}{3}\left(5 + 4^n\right) \equiv 3, 7, 3, 7, \ldots \pmod{10}$	**A1**	2.1	**OR M1** for noting the terms of the sequence mod 10 so that final **A1** can be earned also
			since hcf(3, 10) = 1	**A1**	2.2a	
			and 3, 7 are quadratic non-residues modulo 10, so $H_n \neq$ a square	**A1**	2.4	
				[6]		

Question			Answer	Marks	AOs	Guidance
9	(a)		For constant sequence, set $2 = \frac{k}{6+2}$ (from $u_2 = u_1$) $\Rightarrow k = 16$	M1 A1 [2]	1.1a 1.1	
	(b)		$u_3 = \frac{k}{6+\frac{k}{8}} = \frac{8k}{48+k}$ equated to $u_1 = 2$ Solving for $k \Rightarrow 96 + 2k = 8k \Rightarrow k = 16$ But this is the condition for $\{u_n\}$ constant, so the sequence is never periodic with period 2	M1 A1 A1 [3]	3.1a 1.1 2.3	u_3 correct (simplified) and set = 2 Correct conclusion stated with supporting reason
	(c)		$u_4 = \frac{k}{6+\frac{8k}{48+k}} = \frac{k(k+48)}{288+14k}$, $u_5 = \frac{k}{6+\frac{k(k+48)}{288+14k}} = \frac{k(14k+288)}{k^2+132k+1728}$ For $\{u_n\}$ periodic, period 4: $u_5 = u_1$ $\Rightarrow 14k^2 + 288k = 2k^2 + 264k + 3456$ $12(k^2 + 2k - 288) = 0 \Rightarrow (k - 16)(k + 18) = 0$ $k = -18$	M1 A1 M1 M1 A1 [5]	1.1 1.1 3.1a 1.1 2.2a	u_4 and u_5 attempted in terms of k At least u_4 correct (simplified) u_5 must have been worked out and an algebraic expression equated to 2 Solving a quadratic eqn. in k Correct (single) answer only
10	(a)		The section $x = 0$, i.e. $z = f(0, y)$ drawn The section $y = 0$, i.e. $z = f(x, 0)$ drawn	B1 B1 [2]	3.4 3.4	$\cup$-shaped parabola in y-z plane (axes labelled) $\cap$-shaped parabola in x-z plane (axes labelled)
	(b)		Suggestion $z = ay^2 - bx^2$ $(-0.25 \le x \le 0.25, \ -0.3 \le y \le 0.3)$ $z = 0.27$ when $x = 0, y = (\pm)\ 0.3$ $\Rightarrow a = 3$ $z = -0.4$ when $y = 0, x = (\pm)\ 0.25 \Rightarrow b = 6.4$	M1 * A1 M1 *dep A1 A1 [5]	3.3 1.1 3.4 1.1 2.2a	**M** for quadratic (only) terms in both x and y Details, including signs of coefficients (Domain not required) Use of "boundary" conditions to evaluate a, b (at least one fully attempted)

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