Model Solution

OCR
Oxford Cambridge and RSA

Monday 24 June 2019 – Morning

A Level Further Mathematics A

Y545/01 Additional Pure Mathematics

Time allowed: 1 hour 30 minutes

You must have:
- Printed Answer Booklet
- Formulae A Level Further Mathematics A

You may use:
- a scientific or graphical calculator

* 7 7 7 2 3 9 8 2 0 9 *

INSTRUCTIONS
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Answer **all** the questions.
- **Write your answer to each question in the space provided in the Printed Answer Booklet.** If additional space is required, you should use the lined page(s) at the end of the Printed Answer Booklet. The question number(s) must be clearly shown.
- You are permitted to use a scientific or graphical calculator in this paper.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question.
- The acceleration due to gravity is denoted by $g\,\text{m}\,\text{s}^{-2}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION
- The total mark for this paper is **75**.
- The marks for each question are shown in brackets **[ ]**.
- **You are reminded of the need for clear presentation in your answers.**
- The Printed Answer Booklet consists of **12** pages. The Question Paper consists of **4** pages.

Answer **all** the questions.

1 The sequence $\{u_n\}$ is defined by $u_0 = 2,\ u_1 = 5$ and $u_n = \dfrac{1+u_{n-1}}{u_{n-2}}$ for $n \geqslant 2$.

Prove that the sequence is periodic with period 5. **[4]**

$u_2 = \dfrac{1+5}{2} = 3$

$u_3 = \dfrac{1+3}{5} = 0.8$

$u_4 = \dfrac{1+0.8}{3} = 0.6$

$u_5 = \dfrac{1+0.6}{0.8} = 2$

$u_6 = \dfrac{1+2}{0.6} = 5$

$u_0 = u_5$ and $u_1 = u_6$

$\therefore$ periodicity, period 5

2 A surface has equation $z = f(x, y)$ where $f(x, y) = x^2 \sin y + 2y\cos x$.

(a) Determine f_x, f_y, f_{xx}, f_{yy}, f_{xy} and f_{yx} . **[5]**

(b) **(i)** Verify that z has a stationary point at $\left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi^2\right)$. **[3]**

(ii) Determine the nature of this stationary point. **[3]**

a) $f_x = 2x\sin y - 2y\sin x$ $\quad f_y = x^2\cos y + 2\cos x$

$f_{xx} = 2\sin y - 2y\cos x$ $\quad f_{yy} = -x^2 \sin y$

$f_{xy} = f_{yx} = 2x\cos y - 2\sin x$

bi) when $x = y = \frac{\pi}{2}$, $f_x = f_y = 0$

$\therefore$ stationary point.

$z = \left(\left(\frac{\pi}{2}\right)^2 \times 1\right) + \left(2 \times \frac{\pi}{2} \times 0\right) = \frac{\pi^2}{4}$

bii) $|H| = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & -2 \\ -2 & -\frac{\pi^2}{4} \end{vmatrix} = \left(-\frac{\pi^2}{4} \times 2\right) - \left(-2 \times -2\right) = -\frac{\pi^2}{2} - 4$

As $-\frac{\pi^2}{2} - 4 < 0$, $|H| < 0$ $\therefore$ saddle-point.

3 **(a)** Solve $7x \equiv 6 \pmod{19}$. **[2]**

(b) Show that the following simultaneous linear congruences have no solution.

$x \equiv 3 \pmod 4$, $x \equiv 4 \pmod 6$. **[2]**

a) $7x \equiv 6 \equiv 25 \equiv 44 \equiv 63 \ldots$

$x \equiv 9 \pmod{19}$

b) when $x = 3 \pmod 4$, x is odd

meanwhile

when $x = 4 \pmod 6$, x is even

$\therefore$ there are no solutions.

4 **(a)** Solve the second-order recurrence relation $T_{n+2} + 2T_n = -87$ given that $T_0 = -27$ and $T_1 = 27$. **[8]**

(b) Determine the value of T_{20}. **[2]**

a) PS: $T_n = -29$

CS: $\lambda^2 + 2 = 0$

$\lambda^2 = -2$

$\lambda = \pm i\sqrt{2}$

$T_n = A(i\sqrt{2})^n + B(-i\sqrt{2})^n$

GS: $T_n = A(i\sqrt{2})^n + B(-i\sqrt{2})^n - 29$

As $T_0 = -27$

$T_0 = -27 = A + B - 29$

$A + B = 2$ — ①

As $T_1 = 27$

$T_1 = 27 = Ai\sqrt{2} - Bi\sqrt{2} - 29$

$A - B = -28i\sqrt{2}$ — ②

① + ②:

$A + B + A - B = 2 - 28i\sqrt{2}$

$2A = 2 - 28i\sqrt{2}$

$A = 1 - 14i\sqrt{2}$

$B = 1 + 14i\sqrt{2}$

$\therefore T_n = (1 - 14i\sqrt{2})(i\sqrt{2})^n + (1 + 14i\sqrt{2})(-i\sqrt{2})^n - 29$

b) when $n=20$

$T_{20} = (1-14i\sqrt{2})(i\sqrt{2})^{20} + (1+14i\sqrt{2})(-i\sqrt{2})^{20} - 29$

$= (1-14i\sqrt{2}) \times 1024 + (1+14i\sqrt{2}) \times 1024 - 29$

$= 1024 + 1024 - 29$

$= 2019$

5 The group G consists of a set S together with $\times_{80}$, the operation of multiplication modulo 80. It is given that S is the smallest set which contains the element 11.

(a) By constructing the Cayley table for G, determine all the elements of S. **[5]**

The Cayley table for a second group, H, also with the operation $\times_{80}$, is shown below.

$\times_{80}$	1	9	31	39
1	1	9	31	39
9	9	1	39	31
31	31	39	1	9
39	39	31	9	1

(b) Use the two Cayley tables to explain why G and H are not isomorphic. **[2]**

(c) **(i)** List
- all the proper subgroups of G,
- all the proper subgroups of H. **[3]**

(ii) Use your answers to **(c) (i)** to give another reason why G and H are not isomorphic. **[1]**

a) $11^2 = 121 \equiv 41 \pmod{80} \Rightarrow 41 \in S$

$11 \times 41 = 451 \equiv 51 \pmod{80} \Rightarrow 51 \in S$

$41^2 \equiv 1 \pmod{80} \Rightarrow 1 \in S$

$\times_{80}$	1	11	41	51
1	1	11	41	51
11	11	41	51	1
41	41	51	1	11
51	51	1	11	41

b) · Order of elements of G: 1, 4, 2, 4
· Order of elements of H: 1, 2, 2, 2

OR

· G is a cyclic group of order 4.
· H is a Klein - 4 group.

OR

· G has an element of order 4.
· H has all non-identity elements of order 2.

ci) G has proper subgroup $\{1, 41\}$
H has proper subgroups $\{1, 9\}$, $\{1, 31\}$ and $\{1, 39\}$.

cii) H and G have different structures as they have differing numbers of proper subgroups.

6 (a) For the vectors $\mathbf{p} = \begin{pmatrix}1\\2\\3\end{pmatrix}$, $\mathbf{q} = \begin{pmatrix}3\\1\\-1\end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix}2\\-4\\5\end{pmatrix}$, calculate

- $\mathbf{p} \bullet \mathbf{q} \times \mathbf{r}$,
- $\mathbf{p} \times (\mathbf{q} \times \mathbf{r})$,
- $(\mathbf{p} \times \mathbf{q}) \times \mathbf{r}$. **[6]**

(b) State whether the vector product is associative for three-dimensional column vectors with real components. Justify your answer. **[1]**

It is given that **a**, **b** and **c** are three-dimensional column vectors with real components.

(c) Explain geometrically why the vector $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ must be expressible in the form $\lambda\mathbf{b} + \mu\mathbf{c}$, where λ and μ are scalar constants. **[2]**

It is given that the following relationship holds for **a**, **b** and **c**.

$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \bullet \mathbf{c})\,\mathbf{b} - (\mathbf{a} \bullet \mathbf{b})\,\mathbf{c}$ (*)

(d) Find an expression for $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ in the form of (*). **[3]**

a) $p \cdot q = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 3 & 1 & -1 \end{vmatrix} = \begin{pmatrix} -2-3 \\ -(-1-9) \\ 1-6 \end{pmatrix} = \begin{pmatrix} -5 \\ 10 \\ -5 \end{pmatrix}$

• $p \cdot q \times r = \begin{pmatrix} -5 \\ 10 \\ -5 \end{pmatrix} \times \begin{pmatrix} 2 \\ -4 \\ 5 \end{pmatrix} = -10 - 40 - 25 = -75$

$(q \times r) = \begin{vmatrix} i & j & k \\ 3 & 1 & -1 \\ 2 & -4 & 5 \end{vmatrix} = \begin{pmatrix} 1 \\ -17 \\ -14 \end{pmatrix}$

• $p \times (q \times r) = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 1 & -17 & -14 \end{vmatrix} = \begin{pmatrix} 23 \\ 17 \\ -19 \end{pmatrix}$

$p \times q = \begin{pmatrix} -5 \\ 10 \\ -5 \end{pmatrix}$

• $(p \times q) \times r = \begin{vmatrix} i & j & k \\ -5 & 10 & -5 \\ 2 & -4 & 5 \end{vmatrix} = \begin{pmatrix} 30 \\ 15 \\ 0 \end{pmatrix}$

b) No since $p \times (q \times r) \neq (p \times q) \times r$

c) $\underline{b} \times \underline{c} = \underline{n}$ is normal to the plane containing $\underline{b}, \underline{c}$.

Then $\underline{a} \times \underline{n}$ is perpendicular to this normal and $\therefore$ in the plane of $\underline{b}, \underline{c}$ so that $\underline{a} \times (\underline{b} \times \underline{c})$ is of the form $\lambda \underline{b} + \mu \underline{c}$ for scalar constants λ and μ.

d) $$(\underline{a} \times \underline{b}) \times \underline{c} = -\underline{c} \times (\underline{a} \times \underline{b})$$
$$= -[(\underline{c}\cdot\underline{b})\underline{a} - (\underline{c}\cdot\underline{a})\underline{b}]$$
$$= (\underline{c}\cdot\underline{a})\underline{b} - (\underline{c}\cdot\underline{b})\underline{a}$$

7 The points $P\left(\frac{1}{2}, \frac{13}{24}\right)$ and $Q\left(\frac{3}{2}, \frac{31}{24}\right)$ lie on the curve $y = \frac{1}{3}x^3 + \frac{1}{4x}$.

The area of the surface generated when arc PQ is rotated completely about the x-axis is denoted by A.

(a) Find the exact value of A. Give your answer as a rational multiple of π. **[4]**

Student X finds an approximation to A by modelling the arc PQ as the straight line segment PQ, then rotating this line segment completely about the x-axis to form a surface.

(b) Find the approximation to A obtained by student X. Give your answer as a rational multiple of π. **[4]**

Student Y finds a second approximation to A by modelling the original curve as the line $y = M$, where M is the mean value of the function $f(x) = \frac{1}{3}x^3 + \frac{1}{4x}$, then rotating this line completely about the x-axis to form a surface.

(c) Find the approximation to A obtained by student Y. Give your answer correct to four decimal places. **[4]**

a) $$\frac{dy}{dx} = x^2 - \frac{1}{4x^2}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(x^2 + \frac{1}{4x^2}\right)^2$$

$$A = 2\pi\int\left(\frac{x^3}{3} + \frac{1}{4x}\right)\left(x^2 + \frac{1}{4x^2}\right)dx = 2\pi\int \frac{x^5}{3} + \frac{x}{3} + \frac{1}{16x^3}\,dx$$
$$= 2\pi\left[\frac{x^6}{18} + \frac{x^2}{6} - \frac{1}{32x^2}\right]_{0.5}^{1.5} = 2\pi\left(\frac{1145}{576} - \left(\frac{-95}{576}\right)\right)$$

a continued) $= \dfrac{155\pi}{72}$

b) $y = \frac{3}{4}x + \frac{1}{6}$ which cuts the x-axis at $\left(-\frac{2}{9}, 0\right)$

Distance between $\left(-\frac{2}{9}, 0\right)$ and $\left(\frac{1}{2}, \frac{13}{24}\right)$:

$$\sqrt{\left(-\frac{2}{9} - \frac{1}{2}\right)^2 + \left(0 - \frac{13}{24}\right)^2} = \frac{65}{72}$$

Distance between $\left(-\frac{2}{9}, 0\right)$ and $\left(\frac{3}{2}, \frac{31}{24}\right)$:

$$\sqrt{\left(-\frac{2}{9} - \frac{3}{2}\right)^2 + \left(0 - \frac{31}{24}\right)^2} = \frac{155}{72}$$

Surface area $= \pi\left(\left(\frac{31}{24} \times \frac{155}{72}\right) - \left(\frac{13}{24} - \frac{65}{72}\right)\right)$

$= \dfrac{55\pi}{24}$

OR

$y = \frac{3}{4}x + \frac{1}{6}$

Surface area $= 2\pi \int \left(\frac{3}{4}x + \frac{1}{6}\right)\left(1 + \left(\frac{3}{4}\right)^2\right) dx$

$= \dfrac{55\pi}{24}$

c) $M = \frac{1}{1.5-0.5}\int_{0.5}^{1.5} \frac{x^3}{3} + \frac{1}{4x}\,dx$

$= \left[\frac{x^4}{12} + \frac{1}{4}\ln x\right]_{0.5}^{1.5}$

$= \frac{5}{12} + \frac{1}{4}\ln 3$

Surface area of cylinder (curved) $= 2\pi rh$

$= 2\pi\left(\frac{5}{12} + \frac{1}{4}\ln 3\right)$

$= 4.3437$ (4dp)

8 In this question you must show detailed reasoning.

(a) Prove that $2(p-2)^{p-2} \equiv -1 \pmod{p}$, where p is an odd prime. **[4]**

(b) Find two odd prime factors of the number $N = 2\times 34^{34} - 2^{15}$. **[7]**

a) $2(p-2)^{p-2} \equiv 2(-2)^{p-2} \pmod{p} = -(-2)^{p-1}$

$= -1 \pmod{p}$

(as highest common factor $= ((-1)2, p) = 1$ (as required)

b) mod 3, $N \equiv 2 \times 1^{34} - 2^{15}$

$\equiv 2 \times 1 - 2 = 0$

since $2^{odd} = 2 \pmod{3}$

and $2^{even} = 1 \pmod{3}$

$2 \times 34^{34} - 2^{15} = 2^{35} \times 17^{34} - 2^{15} =$

$2^{15}(2^{20} \times 17^{34} - 1)$

$(2^{20} \times 17^{34} - 1) = (2^{10} \times 17^{17} - 1)(2^{10} \times 17^{17} + 1)$

since $2 \times 17^{17} = -1 \pmod{19}$

$\therefore\ -2^{9} - 1 = -513 = -19 \times 27 = 0 \pmod{19}$

N is a multiple of 19

$N = 2^{15} \times 3 \times 19 \times 389 \times \ldots$