Pearson Edexcel Level 3

GCE Mathematics

Advanced

Paper 2: Pure Mathematics

PMT Mock 1

	Paper Reference(s)
Time: 2 hours	**9MA0/02**

You must have:
Mathematical Formulae and Statistical Tables, calculator

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use black ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 16 questions in this paper. The total mark is 100.
- The marks for each question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.
- If you change your mind about an answer, cross it out and put your new answer and any working underneath.

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1. $$y = \sqrt{(2^x + x)}$$

a. Complete the table below, giving the values of *y* to 3 decimal places.

x	0	0.2	0.4	0.6	0.8	1
y	1	1.161	1.311	1.455	1.594	1.732

(1)

b. Use the trapezium rule with all the values of *y* from your table to find an approximation for the value of

$$\int_0^1 \sqrt{(2^x + x)} \ \ dx$$

giving your answer to 3 significant figures.

The trapezium rule states that:

$$\int_a^b y\,dx \approx \frac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})\}, \text{ where } h = \frac{b-a}{n}$$

$$h = \frac{1-0}{5} = 0.2$$

$$\int_0^1 \sqrt{2^x + x}\ dx \approx \frac{1}{2}(0.2)\big((1 + 1.732) + 2(1.161 + 1.311 + 1.455 + 1.594)\big) = 1.377$$

B1 Uses a strip width of 0.2 units. This may be embedded in the trapezium formula.

e.g. $\frac{0.2}{2}\{\dots\dots\dots\}$

M1 Uses the correct form of the bracket within the trapezium rule.

A1 awrt 1.38

(3)

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Using your answer to part (b) and making your method clear, estimate

c. $\int_0^{0.5} \sqrt{(2^{2x} + 2x)}\ \mathrm{d}x$

$u = 2x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2 \quad \Rightarrow \mathrm{d}x = \frac{\mathrm{d}u}{2}$

$\int_0^{0.5} \sqrt{(2^{2x} + 2x)}\ \mathrm{d}x = \int_0^{1} \sqrt{(2^{u} + u)}\ \frac{\mathrm{d}u}{2}$

$u = 2x \quad \Rightarrow x = 0.5, u = 1 \quad x = 0, u = 0$

$\frac{1}{2}\int_0^{1} \sqrt{(2^{u} + u)}\ \mathrm{d}u = \frac{1}{2} \times 1.38 = 0.689$

(2)

M1 Attempts to answer to (b) $\times \frac{1}{2}$

A1 0.69

(Total for Question 1 is 6 marks)

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2.

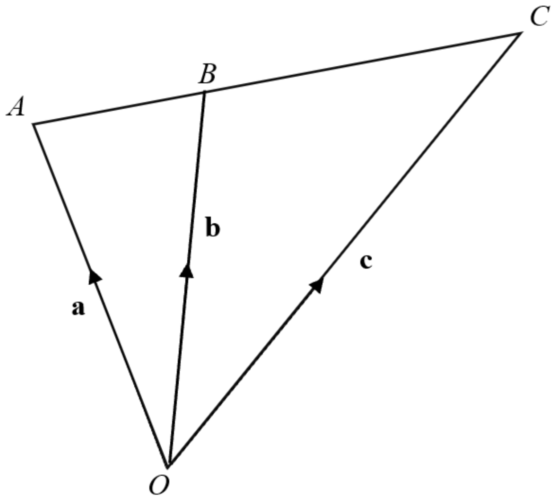

Figure 1

Figure 1 shows a triangle OAC where OB divides AC in the ratio 2: 3.

Show that $\mathbf{b} = \frac{1}{5}(3\mathbf{a} + 2\mathbf{c})$

$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}, \overrightarrow{AC} = \mathbf{c} - \mathbf{a}$ and $\overrightarrow{BC} = \mathbf{c} - \mathbf{b}$

$\overrightarrow{AB} = \frac{2}{5}\overrightarrow{AC}$

$\mathbf{b} - \mathbf{a} = \frac{2}{5}(\mathbf{c} - \mathbf{a}) \Rightarrow 5\mathbf{b} - 5\mathbf{a} = 2(\mathbf{c} - \mathbf{a}) \Rightarrow 5\mathbf{b} - 5\mathbf{a} = 2\mathbf{c} - 2\mathbf{a} \Rightarrow$

$\mathbf{b} = \frac{1}{5}(3\mathbf{a} + 2\mathbf{c})$

M1 Attempts any two of $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}, \overrightarrow{AC} = \mathbf{c} - \mathbf{a}$ and $\overrightarrow{BC} = \mathbf{c} - \mathbf{b}$

dM1 Uses the given information.

e.g. $\overrightarrow{AB} = \frac{2}{5}\overrightarrow{AC}$ or $\overrightarrow{BC} = \frac{3}{5}\overrightarrow{AC}$

A1 Fully correct work including bracketing leading to the given answer.

(3)

(Total for Question 2 is 3 marks)

3. Use the laws of logarithms to solve the equation

$$2 + \log_2(2x + 1) = 2\log_2(22 - x)$$

$$2 + \log_2(2x + 1) = \log_2(22 - x)^2$$
$$2 = \log_2(22 - x)^2 - \log_2(2x + 1)$$
$$2 = \log_2 \frac{(22 - x)^2}{(2x + 1)}$$
$$4 = \frac{(22 - x)^2}{(2x + 1)}$$
$$0 = x^2 - 52x + 480$$
$$0 = (x - 40)(x - 12)$$
$$x = 40, x = 12$$

We reject $x = 40$ as $22 - 40 < 0$, so is undefined

M1 Uses or states $2\log_2(22 - x) = \log_2(22 - x)^2$

M1 Uses addition (or subtraction) law correctly

M1 Connects 2 with 4 OR 2^2 correctly and proceeds to form a quadratic in x

A1 Correct equation, not involving logs, in any form.

M1 Solves a 3TQ by factorisation or by completing of the square or by correct use of formula,

A1 $x = 12$ only and reject $x = 40$

(6)

(Total for Question 3 is 6 marks)

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4. In the binomial expansion of $(2-kx)^{10}$ where k is a non-zero positive constant.

The coefficient of x^4 is 256 times the coefficient of x^6.

Find the value of k.

x^4 coefficient: ${}^{10}C_4 2^6(-k)^4$

x^6 coefficient: ${}^{10}C_6 2^4(-k)^6$

We have

$${}^{10}C_4 2^6(-k)^4 = 256\,{}^{10}C_6 2^4(-k)^6$$

$$210 \times 64k^4 = 256 \times 16k^6 \Rightarrow 4096k^6 - 64k^4 = 0 \Rightarrow 64k^4(64k^2-1) = 0$$

$$k = 0, k = \frac{1}{8}$$

We reject $k = 0$ as the question states that k is non-zero

M1 For an attempt at the correct coefficient of x^4 and x^6

dM1 For ${}^{10}C_4 2^6(-k)^4 = 256 \times {}^{10}C_6 2^4(-k)^6$ and attempt to find k

A1 $k = \frac{1}{8}$ and no other values.

(3)

(Total for Question 4 is 3 marks)

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5. a. Given that

$$\frac{x^2 - 1}{x + 3} \equiv x + P + \frac{Q}{x + 3}$$

find the value of the constant P and show that $Q = 8$

Multiplying both sides by $(x + 3)$:

$$(x^2 - 1) = x(x + 3) + P(x + 3) + Q$$
$$(x^2 - 1) = x^2 + 3x + Px + 3P + Q$$
$$(x^2 - 1) = x^2 + (3 + P)x + 3P + Q$$

$$3 + P = 0 \Rightarrow P = -3, \quad 3P + Q = -1 \Rightarrow Q = 8$$

M1 Multiplies by $(x + 3)$ and attempts to find values for P and Q

Or attempts to divide $x^2 - 1$ by $x + 3$ and obtains a linear quotient and a constant remainder.

A1 $P = -3\,, Q = 8$

(2)

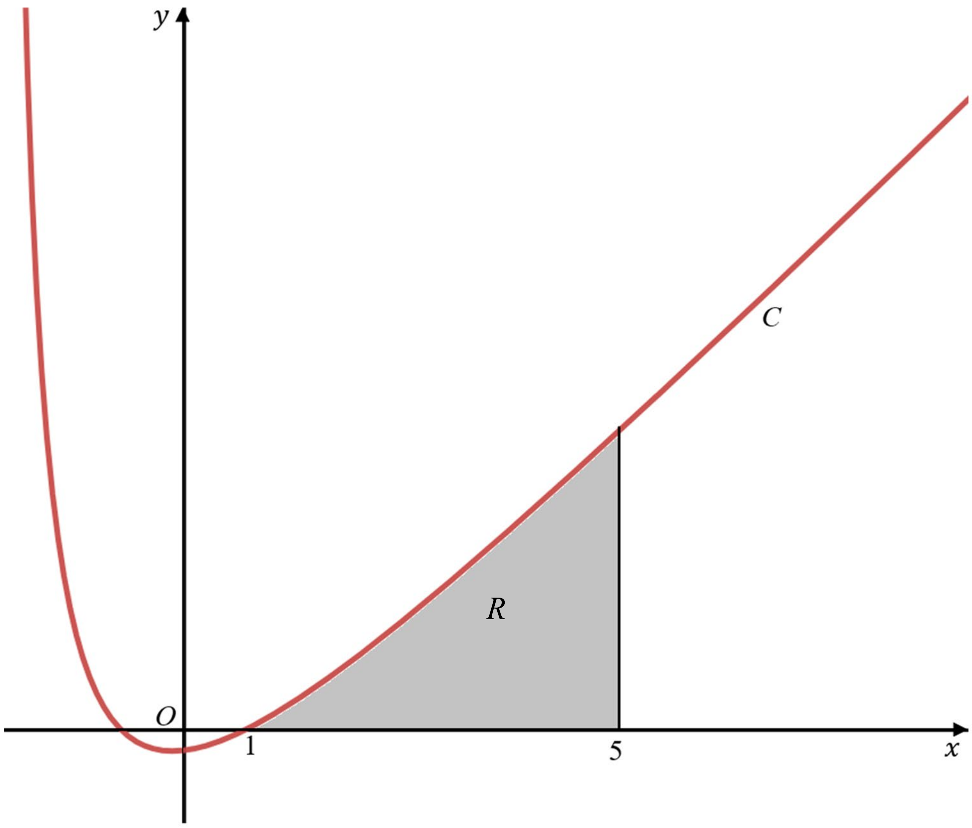

Figure 3

The curve C has equation $y = \mathrm{g}(x)$, where

$$\mathrm{g}(x) = \frac{x^2-1}{x+3} \qquad x > -3$$

Figure 3 shows a sketch of the curve C.

The region R, shown shaded in Figure 4, is bounded by C, the x-axis and the line with equation $x = 5$.

b. Find the exact area of R, writing your answer in the form $a \ln 2$, where a is constant to be found.

$$R = \int_1^5 \frac{x^2-1}{x+3}\, dx$$

$$R = \left[\frac{x^2}{2} - 3x + 8\ln(x+3)\right]_1^5 = \left(\frac{25}{2} - 15 + 8\ln 8\right) - \left(\frac{1}{2} - 3 + 8\ln 4\right) = 8\ln\frac{8}{4}$$

$$= 8\ln 2$$

(4)

b. M1 Integrates an expression of the form $x + P + \frac{Q}{x+3}$ to obtain $\frac{x^2}{2} + Px + k\ln(x+3)$

A1 Correct integration

M1 Substitutes both limits 1 and 5 into an expression $\frac{x^2}{2} + Px + k\ln(x+3)$ and subtracts either way round with fully correct log work to combine two log terms leading to an answer of the form $a \ln b$

A1 $8\ln 2$

(Total for Question 5 is 6 marks)

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6.

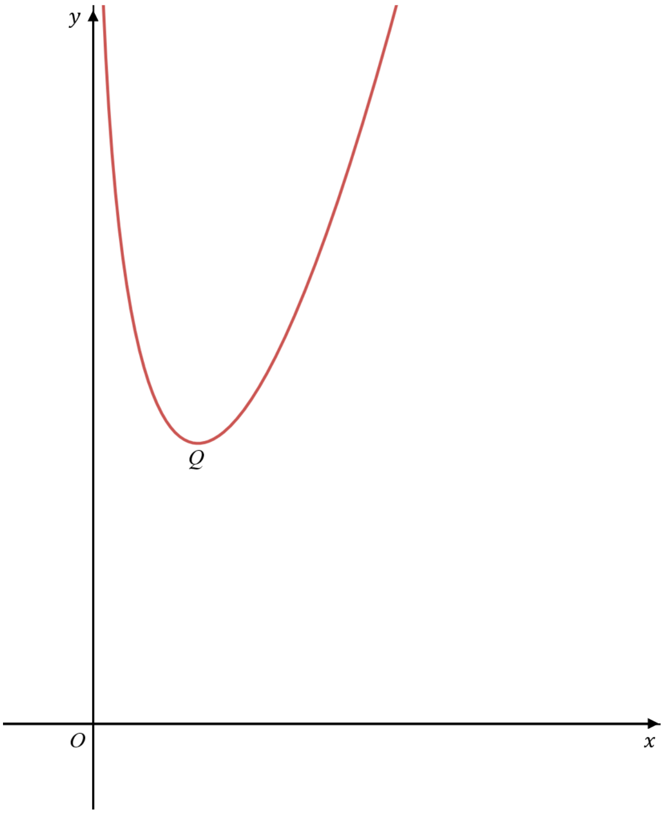

Figure 4

Figure 4 shows a sketch of the curve C with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \frac{2x^2 - x}{\sqrt{x}} - 2\ln\left(\frac{x}{2}\right), \qquad x > 0$$

The curve has a minimum turning point at Q, as shown in Figure 4.

a. Show that $\mathrm{f}'(x) = \frac{6x^2 - x - 4\sqrt{x}}{2x\sqrt{x}}$

$$f'(x) = \frac{d}{dx}\left(\frac{2x^2 - x}{\sqrt{x}}\right) - \frac{d}{dx}\left(2\ln\left(\frac{x}{2}\right)\right)$$

Using the quotient rule on the first term gives:

$$\frac{d}{dx}\left(\frac{2x^2 - x}{\sqrt{x}}\right) = \frac{6x - 1}{2\sqrt{x}}$$

The second term gives:

$$\frac{d}{dx}\left(2\ln\left(\frac{x}{2}\right)\right) = \frac{2}{x}$$

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$$f'(x) = \frac{6x-1}{2\sqrt{x}} - \frac{2}{x} = \frac{6x^2 - x - 4\sqrt{x}}{2x\sqrt{x}}$$

(4)

B1 Differentiates $\ln\frac{x}{2} \rightarrow \frac{1}{x}$

M1 Correct method to differentiate $\frac{2x^2-x}{\sqrt{x}}$

A1 Correct differentiation of $\frac{2x^2-x}{\sqrt{x}}$

A1 Obtains $\frac{dy}{dx} = \frac{6x^2-x-4\sqrt{x}}{2x\sqrt{x}}$

b. Show that the x-coordinate of Q is the solution of

$$x = \sqrt{\frac{x}{6} + \frac{2\sqrt{x}}{3}}$$

We have turning points at $\frac{6x^2-x-4\sqrt{x}}{2x\sqrt{x}} = 0 \Rightarrow 6x^2 - x - 4\sqrt{x} = 0$

$$6x^2 = x + 4\sqrt{x}$$

$$x = \sqrt{\frac{x}{6} + \frac{2\sqrt{x}}{3}}$$

(2)

M1 Sets $6x^2 - x - 4\sqrt{x} = 0$ and writes a line equivalent to $x^2 = \frac{\pm x \pm 4\sqrt{x}}{6}$

A1 Completely correct with all the signs correct.

OR Alternative working backwards

M1 Starts with answer and squares, multiples each side by 6

A1 Completely correct $6x^2 - x - 4\sqrt{x} = 0$ and states $f(x) = 0$

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To find an approximation for the x-coordinate of Q, the iteration formula

$$x_{n+1} = \sqrt{\frac{x_n}{6} + \frac{2\sqrt{x_n}}{3}}$$

is used.

c. Taking $x_0 = 0.8$, find the values of x_1, x_2 and x_3.

Give your answers to 3 decimal places.

$$x_1 = \sqrt{\frac{0.8}{6} + \frac{2\sqrt{0.8}}{3}} = 0.854 \text{ to 3dp}$$

$$x_2 = \sqrt{\frac{0.854}{6} + \frac{2\sqrt{0.854}}{3}} = 0.871 \text{ to 3dp}$$

$$x_3 = \sqrt{\frac{0.871}{6} + \frac{2\sqrt{0.871}}{3}} = 0.876 \text{ to 3dp}$$

(3)

c. M1 An attempt to substitute $x_0 = 0.8$ into the iterative formula. A sight of

A1 Answer which rounds to $x_1 = 0.854$

A1 Both answers which round to $x_2 = 0.871$ and $x_3 = 0.876$

(Total for Question 6 is 9 marks)

7. A curve C has equation $y = \mathrm{f}(x)$.

Given that

- $\mathrm{f}'(x) = 18x^2 + 2ax + b$
- the y-intercept of C is –48
- the point A, with coordinates $(-1,45)$ lies on C

a. show that $a - b = 99$

$$f(x) = \int f'(x)$$
$$f(x) = \int 18x^2 + 2ax + b$$
$$f(x) = \frac{18x^3}{3} + \frac{2ax^2}{2} + bx + c$$
$$f(x) = 6x^3 + ax^2 + bx + c$$

The y-intercept of C is –48 $\Rightarrow c = -48$

The point A, with coordinates $(-1,45)$ lies on $C \Rightarrow 6(-1)^3 + a(-1)^2 + b(-1) - 48 = 45$

$$\Rightarrow -6 + a - b - 48 = 45 \Rightarrow a - b = 99$$

M1 Integrates $\mathrm{f}'(x) = 18x^2 + 2ax + b$

A1 Fully correct integration

B1 Deduces that the constant term is –48

B1 Uses $\mathrm{f}(-1) = 45$ and obtains a linear equation in terms of a and b.

(4)

The tangent to C at the point A has gradient -84.

b. Find the value of a and the value of b.

$$f'(-1) = 18(-1)^2 + 2a(-1) + b = -84 \Rightarrow -2a + b = -84$$

We have two equations:

$a - b = 99$

$-2a + b = -102$

Solving simultaneously: $a = 3, b = -96$

B1 Uses $\mathrm{f}'(-1) = -84$ and obtains a linear equation in terms of a and b.

M1 Attempts to solve simultaneously to get values for both a and b.

A1 $a = 3, b = -96$

(3)

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c. Show that $(2x + 1)$ is a factor of $f(x)$.

The factor theorem states that if $2x + 1$ is a factor of $f(x)$ then $f\left(-\frac{1}{2}\right) = 0$

$$f\left(-\frac{1}{2}\right) = 6\left(-\frac{1}{2}\right)^3 + 3\left(-\frac{1}{2}\right)^2 - 96\left(-\frac{1}{2}\right) - 48 = 0$$

Therefore $(2x + 1)$ is a factor

(2)

M1 Attempts $f(\pm\frac{1}{2})$ or divides by $(2x + 1)$. Look for a constant remainder in the long division.

A1 Obtains a remainder zero and makes a conclusion

e.g. remainder = 0, hence $(2x + 1)$ is a factor.

(Total for Question 7 is 9 marks)

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8.

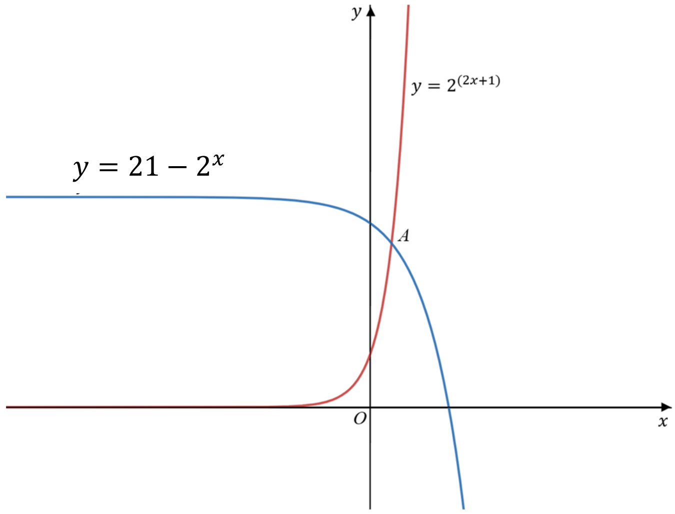

Figure 2

The curves with equation $y = 21 - 2^x$ meet the curve with equation $y = 2^{2x+1}$ at the point A as shown in Figure 2.

Find the exact coordinates of point A.

The curves meet when

$$21 - 2^x = 2^{2x+1}$$

$$2(2^{2x}) + 2^x - 21 = 0$$

Let $y = 2^x$:

$$2y^2 + y - 21 = 0 \Rightarrow y = 3, y = -3.5$$

$$y = 3 \Rightarrow 2^x = 3 \Rightarrow x = \log_2 3$$

$$y = -3.5 \Rightarrow 2^x = -3.5 \Rightarrow x \text{ undefined}$$

Coordinates of A: $(\log_2 3\,, 18)$

B1 Combines the equations $21 - 2^x = 2^{2x+1}$ to reach a correct quadratic equation in 2^x

M1 Solves a quadratic equation of the form $ay^2 + by \pm 21 = 0$ by factorisation or by completion of the square or by correct use of formula.

dM1 Uses logs correctly and proceeds to a value for x from an equation of the form $2^x = k$ where $k > 1$ and attempts to find the corresponding y-value.

A1 Correct solution only $(\log_2 3, 18)$

(4)

(Total for Question 8 is 4 marks)

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9. A cup of tea is cooling down in a room.

The temperature of tea, θ°C, at time t minutes after the tea is made, is modelled by the equation

$$\theta = A + 70e^{-0.025t}$$

where A is a positive constant.

Given that the initial temperature of the tea is 85°C

a. find the value of A.

At time $t = 0, \theta = 85$:

$$85 = A + 70e^{0} \Rightarrow A = 15$$

B1 Substitutes $t = 0, \theta = 85$ into the equation of the model to obtain the value of A.

(1)

b. Find the temperature of the tea 20 minutes after it is made.

At time $t = 20$:

$$\theta = 15 + 70e^{-0.5}$$

$$\theta = 57.46°\text{C}$$

M1 Substitutes $t = 20$ into the given equation with their A to obtain a value for θ

A1 awrt $57.5^{0}C$

(2)

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c. Find how long it will take the tea to cool down to 43°C.

$$43 = 15 + 70e^{-0.025t}$$
$$28 = 70e^{-0.025t}$$
$$e^{-0.025t} = \frac{28}{70}$$
$$t = \frac{\ln\frac{2}{5}}{-0.025} = 36.7$$

M1 Substitutes $\theta = 43$ into the given equation with their A and proceeds to a form

$P = Qe^{-0.025t}$ or $M = Ne^{0.025t}$

A1 For $e^{-0.025t} = \frac{28}{70}$ or $e^{0.025t} = \frac{70}{28}$ or equivalent

M1 Takes ln's correctly to reach $-0.025t = \ln\alpha$, $\alpha > 0$

A1 awrt 36.7 minutes

e.g $t = \frac{\ln\frac{2}{5}}{-0.025} = 36.7$

(4)

(Total for Question 9 is 7 marks)

10. a. Show that

$$\sin 3A \equiv 3\sin A - 4\sin^3 A$$

(4)

Using the angle addition formulae and double angle formulae:

$$\sin(2A + A) = \sin 2A\cos A \pm \cos 2A\sin A$$

$$\sin 3A = 2\sin A\cos A\cos A + (1 - 2\sin^2 A)\sin A$$

$$= 2\sin A\cos^2 A + \sin A - 2\sin^3 A$$

Using $\cos^2 A = 1 - \sin^2 A$

$$\sin 3A = 2\sin A(1 - \sin^2 A) + \sin A - 2\sin^3 A$$

$$\sin 3A \equiv 3\sin A - 4\sin^3 A$$

M1 Attempts to use the identity for $\sin(2A + A) = \sin 2A\cos A \pm \cos 2A\sin A$

dM1 Uses the correct double angle identities for $\sin 2A$ and $\cos 2A$.

ddM1 Reaches an expression in terms of $\sin A$ only by use of $\cos^2 A = 1 - \sin^2 A$

A1 Correct solution only $\sin 3A \equiv 3\sin A - 4\sin^3 A$

b. Hence solve, for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ the equation

$$1 + \sin 3\theta = \cos^2\theta$$

$$\sin 3A \equiv 3\sin A - 4\sin^3 A$$

Using $\cos^2\theta = 1 - \sin^2\theta$:

$$1 + \sin 3\theta = \cos^2\theta \ \Rightarrow 1 + \sin 3\theta = 1 - \sin^2\theta$$

$$1 + \sin 3\theta = 1 - \sin^2\theta \ \Rightarrow \ 1 + 3\sin\theta - 4\sin^3\theta \ = 1 - \sin^2\theta$$

$$4\sin^3\theta - \sin^2\theta - 3\sin\theta = 0 \ \Rightarrow \sin\theta(4\sin^2\theta - \sin\theta - 3) = 0$$

$$\sin\theta(4\sin\theta + 3)(\sin\theta - 1) = 0 \ \Rightarrow \sin\theta = 0, \frac{\pi}{2}, -0.848$$

M1 Attempts to produce an equation just in $\sin\theta$ using both part (a) and the identity $\cos^2\theta = 1 - \sin^2\theta$

dM1 Uses $\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta$ and obtains a cubic equation in $\sin\theta$ and attempts to solve. This could include factorisation or division of a $\sin\theta$ term followed by an attempt to solve the 3 terms quadratic equation in $\sin\theta$ to reach at least one none zero value for $\sin\theta$.

A1 correct solution only $0, \frac{\pi}{2}, -0.848$

(3)

(Total for Question 10 is 7 marks)

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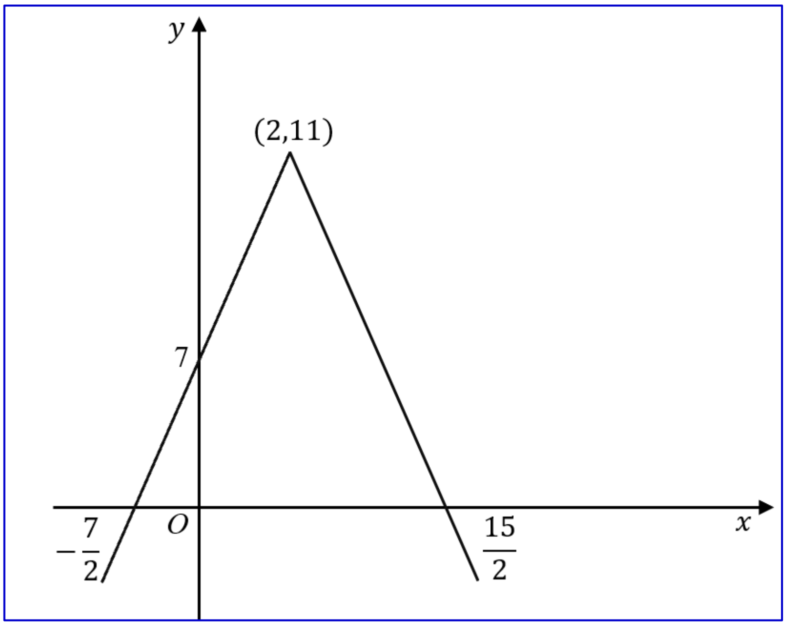
11. a. Sketch the graph of the function with equation

$$y = 11 - 2|2 - x|$$

stating the coordinates of the maximum point and any points where the graph cuts the y-axis.



B1 A ∧-shape with intercepts at $(-\frac{7}{2}, 0)$ and $(\frac{15}{2}, 0)$ or marked $-\frac{7}{2}$ and $\frac{15}{2}$ on the x-axis.

B1 A ∧-shape with intercept at (0,7) or 7 marked on the y-axis

B1 Maximum point at (2,11) that lies in the 1. quadrant.

(3)

b. Solve the equation

$$4x = 11 - 2|2 - x|$$

$4x = 11 + 2(2 - x) \Rightarrow 4x - 11 = 2(2 - x) \Rightarrow x = \frac{5}{2}$

M1 Attempts to solve $4x = 11 + 2(2 - x) \Rightarrow x = \cdots$ Must reach a value for x.

A1 $x = \frac{5}{2}$

(2)

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A straight line l has equation $y = kx + 13$, where k is a constant.

Given that l does not meet or intersect $y = 11 - 2|2 - x|$

c. find the range of possible value of k.

The line $y = kx + 13$ will always pass through $(0,13)$ and at the first point of intersection will intersect at the maximum of $y = 11 - 2|2 - x|$ which is $(2,11)$ from part a.

$$11 = k(2) + 13 \Rightarrow k = -1$$

If $k > -1$, then the lines will not intersect

The lines will intersect if the gradient of $y = kx + 13$ is 'steeper' than that of $y = 11 - 2|2 - x|$, which is 2

So $-1 < k < 2$

M1 Attempts to solve $y = kx + 13$ with their $(2,11)$ to find k or deduces that $k > -1$

A1 Finds that $k = 2$ is a critical value

A1 $-1 < k < 2$

(3)

(Total for Question 11 is 8 marks)

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12.

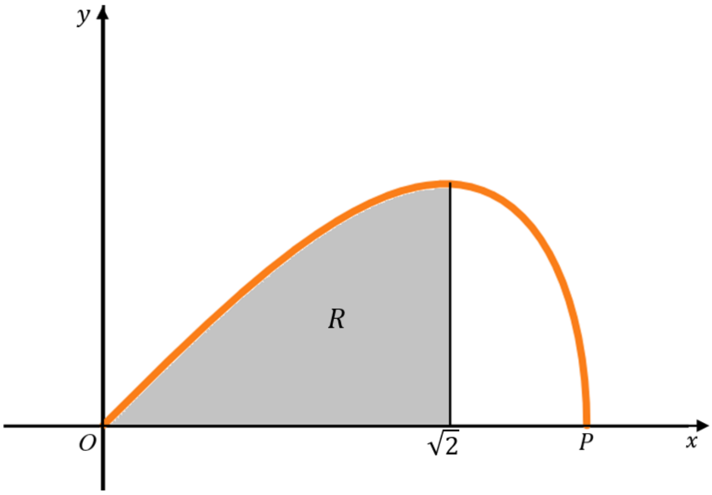

Figure 5

Figure 5 shows part of the curve C with parametric equations

$$x = 2\cos\theta \qquad y = \sin 2\theta \qquad 0 \leq \theta \leq \frac{\pi}{2}$$

The region R, shown shaded in figure 5, is bounded by the curve C, the line $x = \sqrt{2}$ and the x-axis. This shaded region is rotated through 2π radians about the x-axis to form a solid revolution.

a. Show that the volume of the solid of revolution formed is given by the integral.

$$k\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^3\theta\cos^2\theta \quad d\theta$$

where k is a constant.

When a curve, defined by parametric equations, is rotated by 2π radians about the x axis, then the volume is given by:

$$V = \pi\int_a^b y(t)^2(x'(t))\ dt$$

So in this case,

When $x = \sqrt{2} \Rightarrow \theta = \frac{\pi}{4}$, $x = 0 \Rightarrow \theta = \frac{\pi}{2}$

$$V = \pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin 2\theta)^2\ \ (-2\sin\theta)\, d\theta$$

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Using $\sin 2\theta = 2\sin\theta\cos\theta$

$$V = \pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(2\sin\theta\cos\theta)^2\ (-2\sin\theta)\,\mathrm{d}\theta$$

$$= \pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 4\sin^2\theta\cos^2\theta\ (-2\sin\theta)\,\mathrm{d}\theta$$

$$= -8\pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\sin^3\theta\cos^2\theta\ \ \mathrm{d}\theta$$

As we are considering a volume, $k - 8\pi$

M1 Attempts $V = \pi\int y\,\mathrm{d}x = \pi\int y\frac{\mathrm{d}x}{\mathrm{d}\theta}\mathrm{d}\theta$ where $\frac{\mathrm{d}x}{\mathrm{d}\theta} = \pm k\sin\theta$

A1 $\int(\sin 2\theta)^2\ (-2\sin\theta)\,\mathrm{d}\theta$

M1 Attempts to use $\sin 2\theta = 2\sin\theta\cos\theta$ within an integral which may be implied by

B1 Finds correct limits stating $x = 0 \Rightarrow \theta = \frac{\pi}{2}$, $x = \sqrt{2} \Rightarrow \theta = \frac{\pi}{4}$ or a correct value for k.

A1 Achieves printed answer including $\mathrm{d}\theta$ with correct limits and 8π in place with no errors.

(5)

b. Hence, find the exact value for this volume, giving your answer in the form

$p\pi\sqrt{2}$ where p is a constant.

$$u = \cos\theta \;\Rightarrow\; \frac{du}{d\theta} = -\sin\theta$$

$$V = k\int \sin^3\theta\cos^2\theta \; d\theta = k\int \sin^3\theta(-u^2)\frac{du}{\sin\theta} = k\int -u^2\sin^2\theta \; du$$

$$= k\int -u^2(1-u^2)\; du$$

$$= k\left(\frac{u^5}{5} - \frac{u^3}{3}\right)$$

$$8\pi\left[\frac{\cos^5\theta}{5} - \frac{\cos^3\theta}{3}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = 8\pi\left[(0) - \left(\frac{(\frac{\sqrt{2}}{2})^5}{5} - \frac{(\frac{\sqrt{2}}{2})^3}{3}\right)\right] = \frac{7\sqrt{2}}{15}\pi$$

(5)

B1 States $u = \cos\theta \;\Rightarrow\; \frac{du}{d\theta} = -\sin\theta$

M1 Substitutes fully including for $d\theta$ using $u = \cos\theta$ and $\sin^2\theta = \pm 1 \pm \cos^2\theta$ to produce an integral just in terms of u.

M1 Multiplies out to form a polynomial in u and integrates with $u^n \to u^{n+1}$ for at least one of their powers of u.

M1 All methods must have been scored. It is for using the limits 0 and $\frac{\sqrt{2}}{2}$ and subtracting or for using the limits $\frac{\pi}{2}$ and $\frac{\pi}{4}$ if they return to $\cos\theta$.

A1 $V = \frac{7\sqrt{2}}{15}\pi$

(Total for Question 12 is 10 marks)

13. The function g is defined by

$$g(x) = \frac{2e^x - 5}{e^x - 4} \qquad x \neq k, \;\; x > 0$$

where k is a constant.

a. Deduce the value of k.

For the function to be defined, the denominator cannot be equal to zero.

$$e^x - 4 = 0$$

$$x \neq \ln 4$$

B1 Deduces $k = \ln 4$ or $x \neq \ln 4$

(1)

b. Prove that

$$g'(x) < 0$$

For all values of x in the domain of g.

Differentiating $g(x)$ using the quotient rule:

$u = 2e^x - 5 \Rightarrow u' = 2e^x$ and $v = e^x - 4 \Rightarrow v' = e^x$

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(g(x)\big) = \frac{(e^x - 4) \times \alpha e^x - (2e^x - 5) \times \beta e^x}{(e^x - 4)^2}, \qquad \alpha, \beta > 0$$

$$g'(x) = \frac{-3e^x}{(e^x - 4)^2}$$

As $-3e^x < 0$ and $(e^x - 4)^2 > 0$ so $g'(x) < 0$

M1 Attempts to differentiate via the quotient rule/ product rule/ chain rule

A1 $\frac{-3e^x}{(e^x-4)^2}$ or equivalent

A1 Correct solution only. States that as $-3e^x < 0$ and $(e^x - 4)^2 > 0$ so $g'(x) < 0$

(3)

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c. Find the range of values of a for which

$$g(a) > 0$$

$$2e^x - 5 > 0$$

$$a > \ln 4, \quad 0 < a < \ln\frac{5}{2}$$

(2)

M1 Attempts to solve either $2e^x - 5 = 0$ or $e^x - 4 = 0$ or using inequalities

A1 $a > \ln 4, \quad 0 < a < \ln\frac{5}{2}$

(Total for Question 13 is 6 marks)

14. A circle C has equation $x^2 + y^2 - 6x - 14y = 40$.

The line l has equation $y = x + k$, where k is a constant.

a. Show that the x-coordinate of the points where C and l intersect are given by the solutions to the equation

$$2x^2 + (2k - 20)x + k^2 - 14k - 40 = 0$$

Substituting $y = x + k$ into the equation of the circle gives:

$$x^2 + (x + k)^2 - 6x - 14(x + k) = 40$$

$$2x^2 + (2k - 20)x + k^2 - 14k - 40 = 0$$

(2)

M1 Attempts to form an equation with terms of the form x^2, x, k^2 and kx only using $y = x + k$ and $x^2 + y^2 - 6x - 14y = 40$ which must be an appropriate form.

A1 Uses correct and accurate algebra leading to the given solution.

b. Hence find the two values of k for which l is a tangent to C.

l is a tangent to C when they intersect at one point only, meaning that the discriminant of the quadratic in part a must equal zero

$$(2k - 20)^2 - 4(2)(k^2 - 14k - 40) = 0$$

$$k^2 - 8k - 180 = 0$$

$$(k - 18)(k + 10) = 0$$

$$k = 18, k = -10$$

(4)

M1 Attempts to use $b^2 - 4ac = 0$ with $a = 2, b = 2k - 20$ and $c = k^2 - 14k - 40$ and forms a 3TQ equation in terms of k.

A1 Correct quadratic equation in k.

M1 Correct attempt to solve their 3TQ in k.

A1 $k = 18, k = -10$

(Total for Question 14 is 6 marks)

15. An infinite geometric series has first four terms $1 - 2x + 4x^2 - 8x^3 + \cdots$. The series is convergent.

a. Find the set of possible values of x for which the series converges.

For the series to be convergent, $|r| < 1$.

Therefore, $|-2x| < 1$, so $|x| < \frac{1}{2}$

M1 Understands that for the series to be convergent $|r| < 1$ or states $|-2x| < 1$

A1 Correctly concludes that $|x| < \frac{1}{2}$. Accept $-\frac{1}{2} < x < \frac{1}{2}$

(2)

Given that $\sum_{r=1}^{\infty}(-2x)^{r-1} = 8$,

b. calculate the value of x.

From the formula book,

$$S_\infty = \frac{a}{1-r}$$

So

$$S_\infty = \frac{1}{1+2x} = 8$$

$$\frac{1}{8} = 1 + 2x \quad \Rightarrow 2x = -\frac{7}{8} \quad \Rightarrow x = -\frac{7}{16}$$

(3)

M1 Understands to use the sum to infinity formula.

e.g. $\frac{1}{1+2x} = 8$

M1 Attempts to solve for x.

e.g. $\frac{1}{8} = 1 + 2x \quad \Rightarrow 2x = -\frac{7}{8} \quad \Rightarrow x = \cdots$

A1 $x = -\frac{7}{16}$

(Total for Question 15 is 5 marks)

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16. Prove by contradiction that if n^2 is a multiple of 3, n is a multiple of 3.

B1 Assume that there exists a number n that isn't a multiple of 3 yet n^2 is a multiple of 3

M1 States that $m = 3k + 1$ or $m = 3k + 2$ and attempts to square.

Alternatively exist such as that $m = 3k + 1$ or $m = 3k - 1$

M1 States that $m = 3k + 1$ and $m = 3k + 2$ and attempts to square.

A1 Achieves forms that can be argued as to why they are not a multiple of 3

e.g. $m^2 = (3p + 1)^2 = 9p^2 + 6p + 1 = 3(3p^2 + 2p) + 1$

and $m^2 = (3p + 2)^2 = 9p^2 + 12p + 4 = 3(3p^2 + 4p + 1) + 1$

A1 Correct proof which requires

- Correct calculations
- Correct reasons. E.g. $9p^2 + 6p + 1$ or $9p^2 + 12p + 4$ is not a multiple of 3
- Minimal conclusion.

(5)

(Total for Question 16 is 5 marks)