S2 October 2016 IAL (MA)

Q1a) $H_0: p = 0.05$ $\quad X \sim B[30, 0.05]$

$H_1: p > 0.05$ $\quad P(X \geq 4) = 1 - P(X \leq 3)$

$= 1 - 0.9392 = 0.0608$

$0.0608 < 0.10$

$\therefore$ Result is significant.
Reject H_0.
Evidence suggests that the claim is true.

b) For new sample, $Y \sim B[90, 0.05]$

n large, p small. $np = 90 \times 0.05$
$= 4.5$

so $Y \simeq \sim Po(4.5)$

$$\left.\begin{aligned} P(Y \geq 7) &= 0.1689 \\ P(Y \geq 8) &= 0.0866 \end{aligned}\right\} \boxed{Y \geq 8} \text{ is the C.R}$$

Q2a) $F(t) = \begin{cases} 0, & t < 8 \\ \frac{1}{96}\left(74t - \frac{5}{2}t^2 + k\right), & 8 \leq t \leq 12 \\ 1, & t > 12 \end{cases}$

$F(12) = 1$ or $F(8) = 0$

<u>using $F(8) = 0$</u> : $\frac{1}{96}\left(74(8) - \frac{5}{2}(8)^2 + k\right) = 0$ //

$\therefore k = \frac{5}{2}(8)^2 - 74(8) = \boxed{-432}$

b) $\frac{d}{dt}\left(\frac{1}{96}\left(74t - \frac{5}{2}t^2 + -432\right)\right)$

$= \frac{1}{96}(74 - 5t)$ //

$\therefore f(t) = \begin{cases} \frac{1}{96}(74 - 5t), & 8 \leq t \leq 12 \\ 0, & \text{otherwise.} \end{cases}$

c) highest point is at $t = 8$. $\therefore$ $\boxed{\text{mode} = 8}$

d) $F(m) = 0.5$.

$$F(m) = \frac{1}{96}\left(74m - \frac{5}{2}m^2 + -432\right) = 0.5$$

$$74m - 2.5m^2 - 432 = 48$$

$$2.5m^2 - 74m + 480 = 0$$

By Quadratic Formula . . .

$$m = 20 \quad , \quad m = 9.6$$

m has to be in the range
$8 \leq m \leq 12$

so $\boxed{m = 9.6}$

e) $P(\text{required}) = F(9) = \frac{1}{96}\left(74(9) - \frac{5}{2}(9)^2 - 432\right)$

$$= \boxed{0.328}$$

f) $P(\text{lifetime} < 11 \text{ hrs} \mid \text{lifetime} \geq 9 \text{ hrs}) = \dfrac{P(9 < t < 11)}{P(t > 9)}$

$$= \frac{F(11) - F(9)}{1 - F(9)}$$

$$F(11) = \frac{1}{96}\left(74(11) - \frac{5}{2}(11)^2 - 432\right) = \frac{53}{64}$$

$\therefore P(\text{required}) = \dfrac{\frac{53}{64} - 0.328}{1 - 0.328} = \boxed{0.744}$

Q3a) $X \sim B[20, 0.4]$

b) $P(4 \leq X < 9) = P(X < 9) - P(X \leq 3)$

$= P(X \leq 8) - P(X \leq 3)$

$= \boxed{0.5796}$

c) $P(\text{total} > 0)$.

Total no. of points $= 7X - 3(20 - X)$

$= 7X - 60 + 3X$

$= 10X - 60$

$\Rightarrow P(\text{required}) = P(10X - 60 > 0)$

$= P(X > 6) = 1 - P(X \leq 6)$

$= 1 - 0.2500$

$= \boxed{0.7500}$

d) Total points $= 10X - 60$

Variance of total points $= Var(10X - 60)$

$= 10^2 Var(X)$

$= 100 Var(X)$

$Var(X) = np(1-p) = 20(0.4)(1-0.4)$

$= 4.8$

$\therefore$ Var(total points) $= 100 \times 4.8$

$= \boxed{480}$

Q4ai) total area $= 1$.

Area $\Delta = \frac{1}{2} \times 5 \times u = \frac{5u}{2}$

Area $\square = (10.5 - 5) \times u = 5.5u = \frac{11}{2}u$

$\therefore \left(\frac{5}{2} + \frac{11}{2}\right)u = 1$

$8u = 1 \quad \therefore u = \frac{1}{8}$

hence area $\Delta = \frac{5}{16}$

$\therefore \int_0^5 (M\ x)\, dx = \frac{5}{16}$

$$\left[\frac{mx^2}{2}\right]_0^5 = \frac{5}{16}$$

$$\frac{m}{2}(25) = \frac{5}{16}$$

$$m = \frac{5}{16} \times \frac{2}{25} = \boxed{\frac{1}{40}}$$

b) $$E(X) = \int_0^5 (mx^2)\,dx + \int_5^{10.5} (4x)\,dx$$

$$= \frac{1}{40}\left[\frac{x^3}{3}\right]_0^5 + \frac{1}{8}\left[\frac{x^2}{2}\right]_5^{10.5}$$

$$= \frac{1}{40}\left[\frac{125}{3}\right] + \frac{1}{8}\left[\frac{441}{8} - \frac{25}{2}\right]$$

$$= \boxed{\frac{1223}{192}} = 6.37.$$

c) $$\frac{1}{40}\int_0^y [x]\,dx = 0.25$$

where y is the lower Quartile
Area $\Delta = \frac{5}{16} > 0.25$
so LQ lies in first interval.

$$\frac{1}{40}\left[\frac{x^2}{2}\right]_0^y = 0.25$$

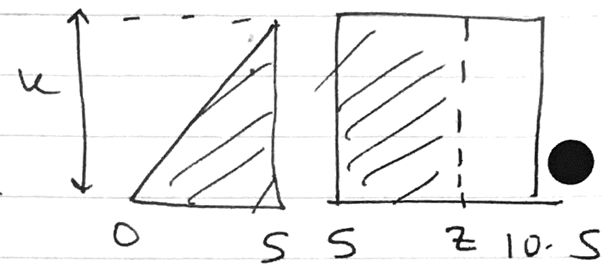
$$\frac{1}{40}\left[\frac{y^2}{2}\right] = 0.25$$

$$\therefore y^2 = 80 \times 0.25 = 20$$

$$\therefore y = \sqrt{20} = LQ.$$

<u>for UQ</u> :

let z be upper Quartile



then shaded area $= 0.75$.

$$\Rightarrow \underbrace{\left(\frac{5}{2} \times \frac{1}{8}\right)}_{\text{Area } \Delta} + \underbrace{(z-5) \times \frac{1}{8}}_{\text{Area}} = 0.75$$

$$\frac{(z-5)}{8} = 0.75 - \frac{5}{16} = \frac{7}{16}$$

$$z - 5 = \frac{7}{2}$$

$$z = 5 + \frac{7}{2} = 8.5 = UQ$$

$$\therefore IQR = 8.5 - \sqrt{20} = \boxed{4.03}$$

Q5a) $L \sim U[20, 40]$

b) $P(27.5 < L < 28.5) = \dfrac{28.5 - 27.5}{20}$

$= \boxed{\dfrac{1}{20}}$

c) $P(\text{required}) = P\left(\left(\frac{L}{4}\right)^2 < 64\right)$

if a string length L is used to form a square then each side will have length $\frac{L}{4}$ // $\therefore$ Area $= \left(\frac{L}{4}\right)^2$ //

$\Rightarrow \quad P\left(\frac{L^2}{16} < 64\right) = P(L^2 < 1024)$

$= \quad P(L < 32) = \dfrac{(32 - 20)}{20}$

$= \boxed{0.60}$

d) Area (bigger) square $= \dfrac{L^2}{16}$ //

Area (smaller) square $= \left(\dfrac{40 - L}{4}\right)^2$ //

the shorter part will have length $40 - L$ $\therefore$ the square made will have side length of $\frac{40 - L}{4}$. hence area $= \left(\frac{40 - L}{4}\right)^2$

Difference in area $= \dfrac{L^2}{16} - \left(\dfrac{(40-L)}{4}\right)^2$

$= \dfrac{L^2}{16} - \dfrac{(40-L)^2}{16}$

$= \dfrac{L^2-(1600-80L+L^2)}{16}$

$= \dfrac{80L-1600}{16} = 5L-100$ //

P(required) $= P(5L-100 > 81)$

$= P(5L > 181)$

$= P\left(L > \frac{181}{5}\right) = P(L > 36.2)$ //

$= \dfrac{40-36.2}{20} = \boxed{0.19}$

Q6a) $X \sim Po(\lambda)$

$$P(X \leq 1) = P(X=0) + P(X=1)$$

$$= \underline{e^{-\lambda} + (e^{-\lambda})(\lambda)}$$

$$= e^{-\lambda}(\lambda+1)$$

b) for 5 weeks . . . $X_2 \sim Po\left(\frac{\lambda}{2}\right)$

$$P(X_2 = 1) = \boxed{\left(e^{-\frac{\lambda}{2}}\right)\left(\frac{\lambda}{2}\right)}$$

ci) for 100 weeks, mean will be large ($\lambda > 10$). so a normal approximation is valid.

ii) $Y \sim Po(10\lambda) \longrightarrow Y \simeq\sim N(10\lambda, 10\lambda)$

(applying c.c) $P(Y < 15) = P(Y < 14.5) = 0.0179$

$$= P\left(Z < \frac{14.5 - 10\lambda}{\sqrt{10\lambda}}\right) = 0.0179$$

$$\therefore \quad P\left(Z < \frac{10\lambda - 14.5}{\sqrt{10\lambda}}\right) = 0.9821$$

but $P(Z < 2.10) = 0.9821$

$$\therefore \quad 2.10 = \frac{10\lambda - 14.5}{\sqrt{10\lambda}}$$

$$10\lambda - (2.10\sqrt{10})\lambda^{\frac{1}{2}} - 14.5 = 0$$

$$10\lambda - 6.64\lambda^{\frac{1}{2}} - 14.5 = 0$$

$$\lambda^{\frac{1}{2}} = 1.58\ldots \quad , \quad \underbrace{\lambda^{\frac{1}{2}} = -0.917\ldots}_{\text{reject, } \lambda > 0.}$$

$$\therefore \lambda = 1.58\ldots^2$$

$$\boxed{= 2.5}$$

Q7a) S could be 2, 3, 4, 5, 6

$$P(S=2) = \frac{5}{8} \times \frac{5}{8} = \frac{25}{64}$$

$$P(S=3) = 2 \times \left[\frac{5}{8}\right]\left[\frac{2}{8}\right] = \frac{5}{16}$$

$$P(S=4) = \left(\frac{2}{8}\right)\left(\frac{2}{8}\right) + 2\left(\frac{5}{8}\right)\left(\frac{1}{8}\right) = \frac{7}{32}$$

$$P(S=5) = 2\left(\frac{1}{8}\right)\left(\frac{2}{8}\right) = \frac{1}{16}$$

$$P(S=6) = \left(\frac{1}{8}\right)\left(\frac{1}{8}\right) = \frac{1}{64}$$

$\therefore$

s	2	3	4	5	6
$P(S=s)$	$\frac{25}{64}$	$\frac{5}{16}$	$\frac{7}{32}$	$\frac{3}{32}$	$\frac{1}{64}$

b) let X = # of scoops ordered by n customes.

$$P(X > n) > 0.99$$

n is the min no. of scoops that can be ordered by n customers.

So $P(X>n) = 1 - P(X=n)$

$$P(X=n) = \left(\frac{5}{8}\right)^n$$ ← all n customers ordering 1 scoop.

$$\therefore P(X>n) = 1 - \left(\frac{5}{8}\right)^n$$

$$\Rightarrow 1 - \left(\frac{5}{8}\right)^n > 0.99$$

$$\left(\frac{5}{8}\right)^n < 0.01$$

$$n \ln\left(\frac{5}{8}\right) < \ln(0.01)$$

(signs change when ÷ by a negative number)

$$n > \frac{\ln(0.01)}{\ln\left(\frac{5}{8}\right)}$$

$\therefore n > 9.798\ldots$ $\therefore$ $\boxed{n_{min} = 10}$