S2 S14 IAL

1. (a) State the conditions under which the Poisson distribution may be used as an approximation to the binomial distribution.
(1)

A farmer supplies a bakery with eggs. The manager of the bakery claims that the proportion of eggs having a double yolk is 0.009
The farmer claims that the proportion of his eggs having a double yolk is more than 0.009

(b) State suitable hypotheses for testing these claims.
(1)

In a batch of 500 eggs the baker records 9 eggs with a double yolk.

(c) Using a suitable approximation, test at the 5% level of significance whether or not this supports the farmer's claim.
(5)

a) $x \sim Po(\lambda)$ from $x \sim B(n,p)$ when n is large and p is small, $np \leq 10$

b) $H_0: \lambda = 0.009$ $\quad x \sim Po(0.009)$
$H_1: \lambda > 0.009$ $\quad x =$ # eggs with a double yolk.

c) $x \sim B(500, 0.009) \approx x \sim Po(4.5)$

$P(x \geq 9)$ $= 1 - P(x \leq 8) = 1 - e^{-4.5} \sum_{0}^{n} \frac{4.5^n}{n!}$
$P(x > 8)$

$= 0.0403$

$< 5\%$ ∴ result is significant
∴ enough evidence to reject null
∴ evidence to support farmer's claim that proportion is higher.

2. The amount of flour used by a factory in a week is Y thousand kg where Y has probability density function

$$f(y) = \begin{cases} k(4 - y^2) & 0 \leqslant y \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Show that the value of k is $\frac{3}{16}$

(4)

Use algebraic integration to find

(b) the mean number of kilograms of flour used by the factory in a week,

(4)

(c) the standard deviation of the number of kilograms of flour used by the factory in a week,

(5)

(d) the probability that more than 1500 kg of flour will be used by the factory next week.

(3)

a) $\int f(y) = 1 \Rightarrow k\int_0^2 4 - y^2 \, dy = 1$

$k\left[4y - \frac{1}{3}y^3\right]_0^2 = 1 \Rightarrow k\left[8 - \frac{8}{3}\right] = 1$

$\Rightarrow k \times \frac{16}{3} = 1 \therefore k = \frac{3}{16}$ #

b) $E(y) = \int y f(y) dy = \int_0^2 \frac{3}{16}y(4 - y^2) dy = \frac{3}{16}\int_0^2 4y - y^3 \, dy$

$= \frac{3}{16}\left[2y^2 - \frac{1}{4}y^4\right]_0^2 = \frac{3}{4}$

c) $E(y^2) = \int_0^2 y^2 f(y) dy = \frac{3}{16}\int_0^2 4y^2 - y^4 dy = \frac{3}{16}\left[\frac{4}{3}y^3 - \frac{1}{5}y^5\right]_0^2$

$E(y^2) = \frac{4}{5}$

$V(y) = E(y^2) - E(y)^2 = \frac{4}{5} - \left(\frac{3}{4}\right)^2 = \frac{19}{80}$

$\therefore Sd_y = \sqrt{\frac{19}{80}} = 0.487$

d) $P(y > 1.5) = 1 - F(1.5) = 1 - \int_0^{1.5} \frac{3}{16}(4 - y^2) dy$

$= 1 - \frac{3}{16}\left[4y - \frac{1}{3}y^3\right]_0^{1.5} = 0.0859$

3. The continuous random variable T is uniformly distributed on the interval $[\alpha, \beta]$ where $\beta > \alpha$

Given that $\mathrm{E}(T) = 2$ and $\mathrm{Var}(T) = \frac{16}{3}$, find

(a) the value of α and the value of β,

(5)

(b) $\mathrm{P}(T < 3.4)$

(2)

$E(T) = 2 \Rightarrow \frac{a+b}{2} = 2 \Rightarrow a+b=4 \Rightarrow a=4-b$

$V(T) = \frac{16}{3} = \frac{(b-a)^2}{12} = \frac{16}{3} \Rightarrow (2b-4)^2 = 64$

$\Rightarrow 2b-4 = 8$

$\therefore b=6 \quad a=-2$

$\alpha=-2, \beta=6$

b) $P(T<3.4) = \frac{3.4+2}{8} = \frac{5.4}{8} = 0.675$

4. Pieces of ribbon are cut to length L cm where $L \sim \mathrm{N}(\mu, 0.5^2)$

(a) Given that 30% of the pieces of ribbon have length more than 100 cm, find the value of μ to the nearest 0.1 cm.

(3)

John selects 12 pieces of ribbon at random.

(b) Find the probability that fewer than 3 of these pieces of ribbon have length more than 100 cm.

(3)

Aditi selects 400 pieces of ribbon at random.

(c) Using a suitable approximation, find the probability that more than 127 of these pieces of ribbon will have length more than 100 cm.

(6)

a) $P(L > 100) = 0.30 \Rightarrow P\left(Z > \frac{100-\mu}{\frac{1}{2}}\right) = 0.30$

% points $\Rightarrow \frac{100-\mu}{\frac{1}{2}} = 0.5244 \quad \therefore \mu = 99.7$

2

b) x = # pieces of ribbon which are > 100cm

$x \sim B(12, 0.3)$

$P(x < 3) = P(x \leq 2)$

$= 0.2528$

c) $\mu = np = 120$ $\quad x \sim B(400, 0.3)$

$\sigma^2 = np(1-p) = 84$ $\quad \Rightarrow \cancel{X} \; x \sim N(120, 84)$

$P(x > 127)$
$P(x \geq 128)$ cc $P(x > 127.5) = P\left(Z > \frac{127.5-120}{\sqrt{84}}\right)$

$\simeq P(Z > 0.8183\ldots) \simeq 0.207$

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5. A company claims that 35% of its peas germinate. In order to test this claim Ann decides to plant 15 of these peas and record the number which germinate.

(a) (i) State suitable hypotheses for a two-tailed test of this claim.

(ii) Using a 5% level of significance, find an appropriate critical region for this test. The probability in each of the tails should be as close to 2.5% as possible.

(4)

(b) Ann found that 8 of the 15 peas germinated. State whether or not the company's claim is supported. Give a reason for your answer.

(2)

(c) State the actual significance level of this test.

(1)

a) x = # peas that germinate
$x \sim B(15, 0.35)$

$H_0 : p = 0.35$
$H_1 : p \neq 0.35$

ii) $P(x \leq L) \simeq 0.025$
$P(x \leq 1) = 0.0142$ *
$P(x \leq 2) = 0.0617$
$\therefore L = 1$

$P(x \geqslant U) \simeq 0.025$
$P(x > U-1) = 1 - P(x \leq U-1) \simeq 0.025$
$\therefore P(x \leq U-1) \simeq 0.975$
$P(x \leq 8) = 0.9578$
$P(x \leq 9) = 0.9876$ * $\therefore U-1=9$
$U=10$

$\therefore$ CR $\{x \leq 1\} \cup \{x \geqslant 10\}$
$\rightarrow \{0 \leq x \leq 1\} \cup \{10 \leq x \leq 15\}$

b) 8 does not fall into the CR $\therefore$ the result is not significant $\therefore$ not enough evidence to reject null $\therefore$ claim is supported.

c) ASL = $\begin{array}{r} 0.0142 \\ +\ 0.0124 \\ \hline \end{array}$ = 0.0266 2.66%.

6. A continuous random variable X has cumulative distribution function $\mathrm{F}(x)$ given by

$$\mathrm{F}(x) = \begin{cases} 0 & x < 0 \\ \dfrac{x^2}{20}(9 - 2x) & 0 \leqslant x \leqslant 2 \\ 1 & x > 2 \end{cases}$$

(a) Verify that the median of X lies between 1.23 and 1.24

(3)

(b) Specify fully the probability density function $\mathrm{f}(x)$.

(3)

(c) Find the mode of X.

(2)

(d) Describe the skewness of this distribution. Justify your answer.

(2)

a) $F(1.23) = 0.495 < 0.5$ Since $F(Q_2) = 0.5$

$F(1.24) = 0.501 > 0.5$ $1.23 < Q_2 < 1.24$

b) $f(x) = \frac{d}{dx}F(x) = \frac{d}{dx}\left(\frac{9x^2}{20} - \frac{x^3}{10}\right) = \frac{9}{10}x - \frac{3x^2}{10}$

$$\therefore \quad f(x) = \begin{cases} \frac{3}{10}(3x - x^2) & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

c) $f'(\text{mode}) = 0 \Rightarrow 3 - 2x = 0 \therefore \text{mode} = 1.5$

d) median < mode $\therefore$ negative skew

7. Flaws occur at random in a particular type of material at a mean rate of 2 per 50 m.

(a) Find the probability that in a randomly chosen 50 m length of this material there will be exactly 5 flaws.

(2)

This material is sold in rolls of length 200 m. Susie buys 4 rolls of this material.

(b) Find the probability that only one of these rolls will have fewer than 7 flaws.

(6)

A piece of this material of length x m is produced.

Using a normal approximation, the probability that this piece of material contains fewer than 26 flaws is 0.5398

(c) Find the value of x.

(8)

a) x = # flaws per 50m $\quad x \sim Po(2)$

$$P(x=5) = \frac{e^{-2} \times 2^5}{5!} = 0.0361$$

b) y = # flaws per 200m $\quad y \sim P(8)$

$$P(y<7) = P(y \leq 6) = 0.3134$$

f = # rolls with <7 flaws

$f \sim B(4, 0.3134) \quad P(x=1) = {}^4C_1\, 0.3134^1\, 0.6866^3$

$$= 0.4058$$

$m = \#$ flaws per x m $\qquad m \sim Po(\lambda^2) \qquad \lambda^2 = \frac{x}{25}$

$\approx m \sim N(\lambda^2, \lambda^2)$

$P(m < 26)$
$P(m \leq 25)$ cc $P(m < 25.5)$

$$\approx P\left(Z < \frac{25.5 - \lambda^2}{\lambda}\right) = 0.5398 \qquad \Phi(0.1) = 0.5398$$

$$\therefore \frac{25.5 - \lambda^2}{\lambda} = 0.1 \Rightarrow \lambda^2 + 0.1\lambda - 25.5 = 0$$

$$\times 10 \quad 10\lambda^2 + \lambda - 255 = 0$$

$$\therefore \lambda = \frac{-1 \pm \sqrt{1 + 4(10)(255)}}{20}$$

$$(10\lambda + 51)(\lambda - 5)$$

$$\therefore \lambda = 5$$

$$\Rightarrow 25 = \frac{x}{25} \qquad \therefore \underline{\underline{x = 625\text{ m}}}$$