S2 IAL Jan 16 (Kprime 2)

1. The manager of a clothing shop wishes to investigate how satisfied customers are with the quality of service they receive. A database of the shop's customers is used as a sampling frame for this investigation.

(a) Identify one potential problem with this sampling frame.
(1)

Customers are asked to complete a survey about the quality of service they receive. Past information shows that 35% of customers complete the survey.

A random sample of 20 customers is taken.

(b) Write down a suitable distribution to model the number of customers in this sample that complete the survey.
(2)

(c) Find the probability that more than half of the customers in the sample complete the survey.
(2)

(a). Not all customer will respond as some customers may have moved areas.

(b) Let X = no. of customers who complete survey.

$X \sim B(20, 0.35)$

(c) $P(X \geq 11) = 1 - P(X \leq 10) = 1 - 0.9468$

$\therefore \quad P(X \geq 11) = 0.0532$

2. The continuous random variable X is uniformly distributed over the interval $[a, b]$

Given that $P(3 < X < 5) = \frac{1}{8}$ and $E(X) = 4$

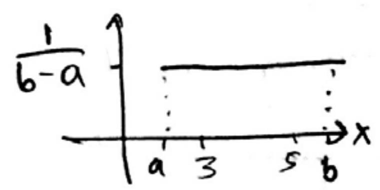
(a) find the value of a and the value of b **(3)**

(b) find the value of the constant, c, such that $E(cX - 2) = 0$ **(2)**

(c) find the exact value of $E(X^2)$ **(3)**

(d) find $P(2X - b > a)$ **(2)**

~~(a)~~



(a) $P(3 < X < 5) = P(X < 5) - P(X < 3)$

$= (5-a)\left(\frac{1}{b-a}\right) - (3-a)\left(\frac{1}{b-a}\right)$

$= \frac{5-a-3+a}{b-a} = \frac{2}{b-a}$

$\Rightarrow \frac{2}{b-a} = \frac{1}{8} \quad \therefore \quad 16 = b-a$

$E(X) = 4 = \frac{1}{2}(a+b) \Rightarrow \quad a+b = 8$

$\therefore \; a+b+b-a = 8+16 = 24$

$\therefore b = 12$

$a = -4$

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Question 2 continued

(b) $E(cX-2) = cE(X)-2 = 4c-2 = 0$

$\Rightarrow c = \frac{1}{2}$

(c) $E(X^2) = Var(X) + [E(X)]^2$

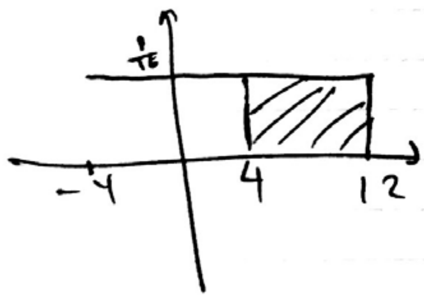
$\therefore E(X^2) = \frac{1}{12}(16)^2 + 4^2$

$E(X^2) = \frac{112}{3}$

(d) $P(2X-b > a) = P(2X-12 > -4)$

$= P(2X > 8) = P(X > 4)$

$= 8 \times \frac{1}{16} = \frac{1}{2}$



$\therefore P(2X-12 > -4) = \frac{1}{2}$

Q2

(Total 10 marks)

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Turn over

3. Left-handed people make up 10% of a population. A random sample of 60 people is taken from this population. The discrete random variable Y represents the number of left-handed people in the sample.

(a) (i) Write down an expression for the exact value of $P(Y \leqslant 1)$

(ii) Evaluate your expression, giving your answer to 3 significant figures.
(3)

(b) Using a Poisson approximation, estimate $P(Y \leqslant 1)$
(2)

(c) Using a normal approximation, estimate $P(Y \leqslant 1)$
(5)

(d) Give a reason why the Poisson approximation is a more suitable estimate of $P(Y \leqslant 1)$
(1)

3(a) $Y \sim Bi(60, 0.1)$

(i) $P(Y \leq 1) = P(Y=0) + P(Y=1)$

~~(ii) $P(Y \leq 1) = (0.9)^{60} + 60($~~

(ii) $P(Y \leq 1) = 0.9^{60} + 60(0.1)(0.9)^{59}$

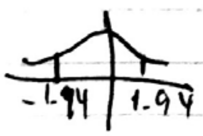
$= 0.0138 \quad (3sf)$

(b) $Y \sim Bi(60, 0.1) \approx X \sim Po(6)$

$P(Y \leq 1) \approx P(X \leq 1) = 0.0174$

(c) $Y \sim Bi(60, 0.1) \approx W \sim N(6, 5.4)$

$P(Y \leq 1) \approx P(W \leq 1.5) = P(Z \leq -1.94)$



$= P(Z \geq 1.94)$

$= 1 - \Phi(1.94)$

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Question 3 continued

$= 1 - 0.9738 = 0.0262$

(d) $n = 60$ $p = 0.1$

$p = 0.1$ is fairly big to yield an accurate normal approximation, therefore Poisson is more suitable since $np \leq 10$ is a suitable parameter.

4. A continuous random variable X has cumulative distribution function

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{4}x & 0 \leqslant x \leqslant 1 \\ \frac{1}{20}x^4 + \frac{1}{5} & 1 < x \leqslant d \\ 1 & x > d \end{cases}$$

(a) Show that $d = 2$ **(2)**

(b) Find $P(X < 1.5)$ **(2)**

(c) Write down the value of the lower quartile of X **(1)**

(d) Find the median of X **(3)**

(e) Find, to 3 significant figures, the value of k such that $P(X > 1.9) = P(X < k)$ **(4)**

4(a) $F(d) = 1$

$\Rightarrow P(X < d) = \frac{1}{20}x^4 + \frac{1}{5}$

$= \frac{1}{20}d^4 + \frac{1}{5} = 1$

~~$\therefore d^4 = 24 \Rightarrow$~~

~~$\therefore d^4 = 16$~~ $\therefore d^4 = (1 - \frac{1}{5}) \times 20 = 16$

$\Rightarrow d = 16^{1/4} = 2$

(b) $P(X < 1.5) = \frac{1}{20}(1.5)^4 + \frac{1}{5}$

$= 29/64$

P 4 6 9 5 0 A 0 1 0 2 4

(c) $LQ = 1$

(d) $1 < \text{median} \leq 2$ let $m = \text{median}$

$\therefore P(X \leq m) = \frac{1}{20}m^4 + \frac{1}{5} = \frac{1}{2}$

$\therefore m^4 = 6$

$\Rightarrow m = 6^{1/4} = 1.57 \text{ (3sf)}$

(e) $P(X > 1.9) = 1 - P(X \leq 1.9)$

$= 1 - F(1.9)$

$= 1 - \frac{1}{20}(1.9)^4 - \frac{1}{5} = 0.148395$

$\therefore P(X < k) = 0.148395$

$\Rightarrow 0 \leq k \leq 1$

$P(X < k) = \frac{1}{4}k = 0.148395$

$\Rightarrow k = 0.594 \text{ (3sf)}$

5. The number of eruptions of a volcano in a 10 year period is modelled by a Poisson distribution with mean 1

(a) Find the probability that this volcano erupts at least once in each of 2 randomly selected 10 year periods.
(2)

(b) Find the probability that this volcano does not erupt in a randomly selected 20 year period.
(2)

The probability that this volcano erupts exactly 4 times in a randomly selected w year period is 0.0443 to 3 significant figures.

(c) Use the tables to find the value of w
(3)

A scientist claims that the mean number of eruptions of this volcano in a 10 year period is more than 1

She selects a 100 year period at random in order to test her claim.

(d) State the null hypothesis for this test.
(1)

(e) Determine the critical region for the test at the 5% level of significance.
(2)

5 let X = no. of eruptions in a 10yr period.

$X \sim Po(1)$

(a) $[P(X \geq 1)]^2 = [1 - P(X = 0)]^2$

$= (0.6321)^2 = 0.39955\ldots$

≈ 0.4

(b) X' = no. of eruptions in 20yr period

$X' \sim Po(2)$

$P(X' = 0) = 0.1353$

Question 5 continued

(c) Let Y = no. of eruptions in a w year period.

~~$Y \sim Po($~~ 10 year period $\Rightarrow$ 1 eruption

1 year period $\Rightarrow$ 0.1 eruption

~~$Y \sim Po(\frac{w}{10})$~~ $Y \sim Po(0.1w)$

~~$P(Y=4)$ $1-P(Y \leq 3) = 0.0443$~~

~~$\therefore P(Y<3) = 0.9557$~~

$P(Y=4) = P(Y \leq 4) - P(Y \leq 3) = 0.0443$

$\therefore P(Y=4) = 0.0744 - 0.0301 = 0.0443$

$\Rightarrow \frac{w}{10} = 8.5 \Rightarrow w = 85$

(d) Let Z = no. of eruptions in a 100 yr period

$Z \sim Po(10)$

$H_0: \lambda = 10$

(e) $H_1: \lambda > 10$ $P(Z \geq c) \leq 0.05$

Critical Region: $Z \geq 16$

$\therefore 1 - P(Z \leq c-1) \leq 0.05$ $\quad c-1 = 15$

$\therefore P(Z \leq c-1) \geq 0.95$ $\quad \Rightarrow c = 16$

6. A continuous random variable X has probability density function

$$f(x) = \begin{cases} ax^2 + bx & 1 \leqslant x \leqslant 7 \\ 0 & \text{otherwise} \end{cases}$$

where a and b are constants.

(a) Show that $114a + 24b = 1$ **(4)**

Given that $a = \frac{1}{90}$

(b) use algebraic integration to find E(X) **(4)**

(c) find the cumulative distribution function of X, specifying it for all values of x **(3)**

(d) find P(X > E(X)) **(2)**

(e) use your answer to part (d) to describe the skewness of the distribution. **(2)**

6 (a). $\int_1^7 f(x)\,\partial x = 1$

$\Rightarrow \int_1^7 ax^2 + bx\,\partial x = \left[\frac{a}{3}x^3 + \frac{bx^2}{2}\right]_1^7$

$= \frac{343}{3}a + \frac{49}{2}b - \frac{a}{3} - \frac{1}{2}b$

$= 1$

$\therefore \left(\frac{343}{3} - \frac{1}{3}\right)a + \left(\frac{49}{2} - \frac{1}{2}\right)b = 1$

$\Rightarrow 114a + 24b = 1$

Question 6 continued

(b) $a = \frac{1}{90}$. $114a + 24b = \frac{19}{15} + 24b = 1$

$\Rightarrow b = -\frac{1}{90}$

$\therefore E(X) = \int_1^7 x f(x)\,dx = \int_1^7 \frac{x^3}{90} - \frac{1}{90}x^2\,dx$

$= \left[\frac{x^4}{360} - \frac{1}{270}x^3\right]_1^7$

$= \frac{2401}{360} - \frac{343}{270} - \frac{1}{360} + \frac{1}{270}$

$= 5.4$

(c) $\cancel{\int x f(x)\,dx =}$

$\int f(x)\,dx = \int \frac{1}{90}x^2 - \frac{1}{90}x\,dx = \frac{1}{270}x^3 - \frac{1}{180}x^2 + C$

$F(7) = 1 \Rightarrow \frac{1}{270}(7)^3 - \frac{1}{180}(7)^2 + C = 1$

$\therefore \frac{539}{540} + C = 1 \Rightarrow C = \frac{1}{540}$

$\therefore F(x) = \begin{cases} 0 & , x < 1 \\ \frac{1}{270}x^3 - \frac{1}{180}x^2 + \frac{1}{540}, & 1 < x \le 7 \\ 1 & , x > 7 \end{cases}$

Q6

(Total 15 marks)

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(d) $P(X > 5.4) = 1 - F(5.4)$

$$= 1 - \frac{5.4^3}{270} + \frac{1}{180}(5.4)^2 - \frac{1}{540}$$

$= \cancel{0.5788}$ $= 0.5769$

(e) $P(X > 5.4) = 0.5769 > 0.5$

$\Rightarrow$ Median > 5.4

$\therefore$ median $>$ mean

$\Rightarrow$ Negative skew.

($P(X >$ a value bigger than 0.54) will yield a value Closer to 0.5. Hence, the median must be greater than 0.54.

$\therefore$ median $>$ mean $\Rightarrow$ -ve skew.)

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7. A fisherman is known to catch fish at a mean rate of 4 per hour. The number of fish caught by the fisherman in an hour follows a Poisson distribution.

The fisherman takes 5 fishing trips each lasting 1 hour.

(a) Find the probability that this fisherman catches at least 6 fish on exactly 3 of these trips.
(6)

The fisherman buys some new equipment and wants to test whether or not there is a change in the mean number of fish caught per hour.

Given that the fisherman caught 14 fish in a 2 hour period using the new equipment,

(b) carry out the test at the 5% level of significance. State your hypotheses clearly.
(6)

7. Let Y = no. of fish caught by fisherman in 1hr period

$Y \sim Po(4)$

In a one hour period, probability that at least 6 fish are caught is

$P(Y \geq 6) = 1 - P(Y \leq 5) = 1 - 0.7851$
$= 0.2149$

Let X = no. of trips in which the fisherman catches at least 6 fish.

$X \sim Bi(5, 0.2149)$

~~$P(X=3) = P(X=0) + P(X=1) + P(X=2) + P$~~

$P(X=3) = {}^5C_3 \times 0.2149^3 \times 0.7851^2$

$= 0.0612$ (3sf)

P 4 6 9 5 0 A 0 2 2 2 4

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Question 7 continued

(b) Let W = no. of fish caught in 2hr period.

$W \sim Po(8)$

$H_0: \lambda = 8$

$H_1: \lambda \neq 8$

$P(W \leq c_1) \leq 0.025 \Rightarrow c_1 = 2$

$P(W \geq c_2) \leq 0.025$

$\therefore 1 - P(W \leq c_2 - 1) \leq 0.025$

$\therefore P(W \leq c_2 - 1) \geq 0.975$

$\Rightarrow c_2 - 1 = 14 \therefore c_2 = 15$

$\therefore$ Critical Regions: $W \leq 2$, $W \geq 15$

Fisherman caught 14 fish.

$W = 14$ is not in critical region.

$\therefore$ Accept H_0, reject H_1.

There is not sufficient evidence to suggest that there is a change in the mean no. of fish caught per hour.