C34 Specimen (MA)

Q1a) $5\cos 2\theta - 12\sin 2\theta \equiv R\cos(2\theta + \alpha) \equiv R\cos 2\theta \cos\alpha - R\sin 2\theta \sin\alpha$

Comparing coefficients: $5 = R\cos\alpha$ ~ ①
$12 = R\sin\alpha$ ~ ②

$\frac{②}{①}$: $\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = \frac{12}{5}$

$\alpha = \tan^{-1}\left(\frac{12}{5}\right) = 67.38°$ //

Finding R: $R = \sqrt{5^2 + 12^2} = 13$ //

$\therefore 5\cos 2\theta - 12\sin 2\theta \equiv \boxed{13\cos(2\theta + 67.38°)}$

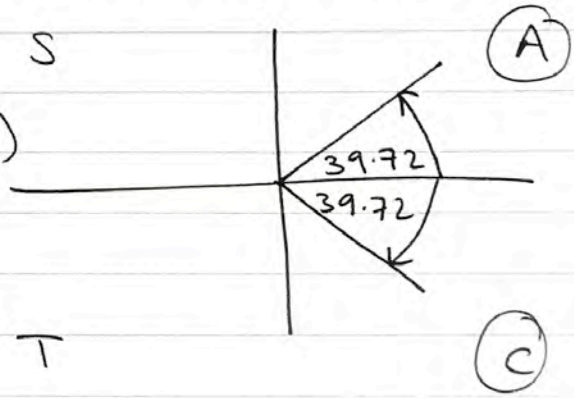
b) $13\cos(2\theta + 67.38°) = 10$

$\cos(2\theta + 67.38°) = \frac{10}{13}$

$2\theta + 67.38° = \cos^{-1}\left(\frac{10}{13}\right) = 39.72°$ //

solving in: $67.38 \leq 2\theta + 67.38° < 427.38$

$2\theta + 67.38° = (360 - 39.72)$
$(360 + 39.72)$



$2\theta + 67.38° = 320.28°,$
$399.72°$

$2\theta = 252.90°, 332.34$

$\boxed{\theta = 126.5°, 166.2°}$

Q2a)

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$3\pi/4$	π
y	0	1.84432	4.81048	8.87207	0

b) $h = \frac{b-a}{n} = \frac{\pi - 0}{4}$ [5 values of x $\therefore$ $n = 4$]

$$\text{Area} \simeq \frac{1}{2} \times \frac{\pi}{4}\left[0+0+2(1.84432 + 4.81048 + 8.87207)\right]$$

$$\simeq \boxed{12.1948}$$

c) $y = e^x (\sin x)^{\frac{1}{2}}$ (PRODUCT RULE)

$$\frac{dy}{dx} = e^x(\sin x)^{\frac{1}{2}} + \frac{1}{2}e^x(\sin x)^{-\frac{1}{2}} \cdot \cos x = 0$$

$$\Rightarrow \quad e^x\sqrt{\sin x} + \frac{e^x \cos x}{2\sqrt{\sin x}} = 0$$

$\xrightarrow{\times 2\sqrt{\sin x}}$: $2e^x \sin x + e^x \cos x = 0$

$$\Rightarrow \quad e^x(2\sin x + \cos x) = 0$$

$e^x \neq 0$, $2\sin x + \cos x = 0$

$\div \cos x$: $2\tan x = -1$ $\therefore$ $\tan x = -\frac{1}{2}$

$$x = \tan^{-1}\left(-\frac{1}{2}\right) = \boxed{2.68}$$

Q3) $\int_0^{\frac{\pi}{2}} \left[e^{\cos x+1}\right] \sin x \, dx$

$$\left[\begin{array}{l} u = \cos x + 1 \\ \frac{du}{dx} = -\sin x \\ dx = \left(\frac{-1}{\sin x}\right) du \end{array}\right]$$

$\Rightarrow \int_2^1 \left[e^u \sin x \times \frac{-1}{\sin x}\right] du$

x	u
0	2
$\frac{\pi}{2}$	1

$u = \cos(0) + 1 = 2$

$u = \cos(\frac{\pi}{2}) + 1 = 1$

$\Rightarrow -\int_2^1 (e^u) \, du$

$\Rightarrow \int_1^2 (e^u) \, du$

$\Rightarrow \left[e^u\right]_1^2 = e^2 - e = \boxed{e(e-1)}$

// $\square$

Q4a) $(2-3x)^{-2} = 2^{-2}\left(1-\frac{3}{2}x\right)^{-2} = \frac{1}{4}\left(1-\frac{3}{2}x\right)^{-2}$

$= \frac{1}{4}\left(1-\frac{3x}{2}\right)^{-2} = \frac{1}{4}\left[1 + 3x + \frac{-2(-3)}{2}\left(-\frac{3x}{2}\right)^2 + \frac{-2(-3)(-4)}{6}\left(-\frac{3x}{2}\right)^3\right]$

$\left(\begin{array}{l} n = -2 \\ x = -\frac{3x}{2} \end{array}\right)$

$= \frac{1}{4}\left[1 + 3x + \frac{27}{4}x^2 + \frac{27}{2}x^3\right]$

$= \boxed{\frac{1}{4} + \frac{3x}{4} + \frac{27}{16}x^2 + \frac{27}{8}x^3}$

b) $f(x) \simeq a + bx\left[\frac{1}{4} + \frac{3x}{4} + \frac{27}{16}x^2 + \frac{27}{8}x^3\right]$

$$\simeq \frac{a}{4} + \frac{bx}{4} + \frac{3ax}{4} + \frac{3bx^2}{4} + \frac{27a}{16}x^2 + \frac{27}{16}bx^3 + \frac{27}{8}ax^3 + \frac{27}{8}bx^4$$

$$\simeq \frac{a}{4} + \left(\frac{b+3a}{4}\right)x + \left(\frac{3b}{4} + \frac{27a}{16}\right)x^2 + \left(\frac{27b}{16} + \frac{27a}{8}\right)x^3 + \ldots$$

<u>coeff of x is 0:</u> $\frac{b+3a}{4} = 0$

$\therefore\ b + 3a = 0 \sim$ 1

<u>coeff of x^2 is $\frac{9}{16}$:</u> $\frac{3b}{4} + \frac{27a}{16} = \frac{9}{16}$

$12b + 27a = 9$

$\div 3\quad 4b + 9a = 3 \sim$ 2

1 $\times 3$: $[3b + 9a = 0]$

$-$2 : $[4b + 9a = 3]$

$-b + 0 = -3$

$\therefore\ \boxed{b = 3} \longrightarrow a = \frac{-3}{3} = \boxed{-1}$

c) x^3 coeff $= \frac{27b}{16} + \frac{27a}{8} = \frac{27(3)}{16} + \frac{27(-1)}{8}$

$$= \boxed{\frac{27}{16}}$$

Q5a) $f(x) = e^{-x} + 2$

$g(x) = 2\ln x$

$fg(x) = e^{-(2\ln x)} + 2 = e^{\ln(x^{-2})} + 2$

$= \boxed{\frac{1}{x^2} + 2}$

b) $f(2x+3) = e^{-(2x+3)} + 2 = 6$

$\Rightarrow \quad e^{-2x-3} = 4$

$\Rightarrow \quad \ln 4 = \ln(e^{-2x-3})$

$\Rightarrow \quad \ln 4 = -2x - 3$

$\Rightarrow \quad 2x = -\ln 4 - 3$

$\Rightarrow \quad x = -\frac{1}{2}\ln 4 - \frac{3}{2} = \boxed{-\ln 2 - \frac{3}{2}}$

c) $y = e^{-x} + 2$

$x \leftrightarrow y$; $x = e^{-y} + 2$

Making y the subject again:

$x - 2 = e^{-y}$

$\ln(x-2) = -y$

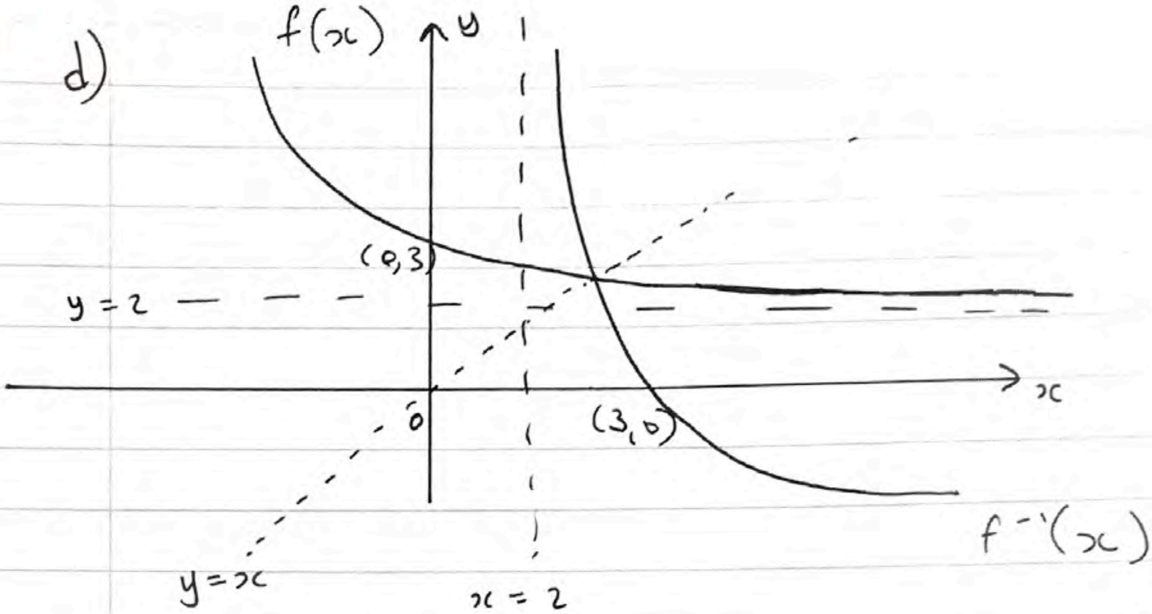
$\therefore \; y = -\ln(x-2)$

$\boxed{y = \ln\left(\frac{1}{x-2}\right)}$

Range of $f(x)$: $y > 2$

so Domain of $f^{-1}(x)$: $\boxed{x > 2}$

d)



Q6a) $\frac{d}{dx}\left(16y^3 + 9x^2y - 54x = 0\right)$

$\Rightarrow 48y^2\frac{dy}{dx} + 18xy + 9x^2\frac{dy}{dx} - 54 = 0$

$\Rightarrow \frac{dy}{dx}\left(48y^2 + 9x^2\right) = 54 - 18xy$

$\Rightarrow \frac{dy}{dx} = \frac{54 - 18xy}{9x^2 + 48y^2}$ //

b) $\frac{dy}{dx} = 0$: $54 - 18xy = 0$

$18xy = 54$

$xy = 3$ //

$\Rightarrow x = \frac{3}{y}$ //

substitute $x = \frac{3}{y}$ into eqn,

$$16y^3 + 9\left(\frac{3}{y}\right)^2 y - 54\left(\frac{3}{y}\right) = 0$$

$$16y^3 + \frac{81}{y} - \frac{162}{y} = 0$$

$$\xrightarrow{\times y} 16y^4 = 81$$

$$\Rightarrow y^4 = \frac{81}{16} \quad \therefore y = \pm\frac{3}{2} \quad \left(= \sqrt[4]{\frac{81}{16}}\right)$$

$$\Rightarrow x = \frac{3}{\pm\frac{3}{2}} \quad \therefore \text{points are: } \boxed{\begin{array}{l}\left(2, \frac{3}{2}\right) \\ \left(-2, -\frac{3}{2}\right)\end{array}}$$

$$x = \pm 2$$

Q7a) $$\text{LHS} = \frac{\cos x}{\sin x} - \frac{\cos 2x}{\sin 2x} = \frac{\cos x}{\sin x} - \frac{\cos 2x}{2\sin x\cos x}$$

$$= \frac{2\cos^2 x}{2\sin x\cos x} - \frac{\cos 2x}{2\sin x\cos x} = \frac{2\cos^2 x - (2\cos^2 x - 1)}{2\sin x\cos x}$$

$$= \frac{1}{2\sin x\cos x} = \boxed{\operatorname{cosec} 2x} \quad \square$$

b) from (a), $\cot x = \operatorname{cosec} 2x + \cot 2x$

$$\therefore \cot\left(\frac{3\theta}{2} + \frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} = \frac{1}{\tan\left(\frac{3\theta}{2} + \frac{\pi}{6}\right)}$$

$$\Rightarrow \tan\left(\frac{3\theta}{2} + \frac{\pi}{6}\right) = \sqrt{3}$$

$$\Rightarrow \tan^{-1}(\sqrt{3}) = \frac{3\theta}{2} + \frac{\pi}{6} = \frac{\pi}{3}$$

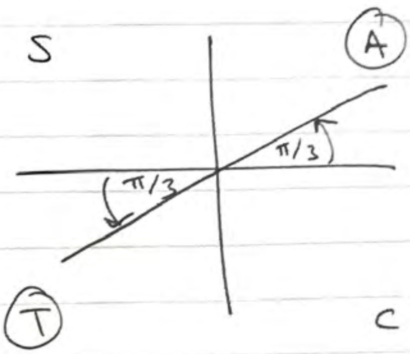
$$\frac{3\theta}{2} + \frac{\pi}{6} = \frac{\pi}{3}$$

<u>solving in :</u> $\frac{\pi}{6} \leq \frac{3\theta}{2} + \frac{\pi}{6} \leq \frac{5\pi}{3}$

$$\frac{3\theta}{2} + \frac{\pi}{6} = \frac{\pi}{3} , \frac{4\pi}{3}$$

$$\frac{3\theta}{2} = \frac{\pi}{6} , \frac{7\pi}{6}$$

$$\boxed{\theta = \frac{\pi}{9} , \frac{7\pi}{9}}$$



Q8a) $h(x) = \frac{2(x^2+5)}{(x+2)(x^2+5)} + \frac{4(x+2)}{(x+2)(x^2+5)} - \frac{18}{(x+2)(x^2+5)}$

$$= \frac{2x^2 + 10 + 4x + 8 - 18}{(x+2)(x^2+5)}$$

$$= \frac{2x^2 + 4x + \cancel{18} - \cancel{18}}{(x+2)(x^2+5)}$$

$$= \frac{2x\cancel{(x+2)}}{\cancel{(x+2)}(x^2+5)} = \boxed{\frac{2x}{x^2+5}}$$

b) $h(x) = 2x(x^2+5)^{-1}$

PRODUCT RULE : $h'(x) = 2(x^2+5)^{-1} - 2x(x^2+5)^{-2}(2x)$

$$h'(x) = \frac{2}{x^2+5} - \frac{4x^2}{(x^2+5)^2}$$

$$= \frac{2(x^2+5)}{(x^2+5)^2} - \frac{4x^2}{(x^2+5)^2}$$

$$= \frac{2x^2+10-4x^2}{(x^2+5)^2}$$

$$= \boxed{\frac{10-2x^2}{(x^2+5)^2}}$$

c) to find max point : $\frac{dy}{dx} = 0$ //

$$h'(x) = \frac{10-2x^2}{(x^2+5)^2} = 0$$

$$10 - 2x^2 = 0$$

$$5 = x^2$$

$x = \pm\sqrt{5}$ // $\rightarrow$ $x = \sqrt{5}$ // $(x > 0)$

at $x = \sqrt{5}$, $y = \frac{2\sqrt{5}}{5+5} = \frac{\sqrt{5}}{5}$ //

So range : $0 \leq y \leq \frac{\sqrt{5}}{5}$

Q9a) $\begin{pmatrix} 2+\lambda \\ 3+2\lambda \\ \lambda-4 \end{pmatrix} = \begin{pmatrix} 5\mu \\ 9 \\ 2\mu-3 \end{pmatrix}$ — 1 — 2 — 3

1 : $\lambda = 5\mu - 2$

$\hookrightarrow$ 3 : $5\mu - 6 = 2\mu - 3$

$3\mu = 3 \quad \therefore \mu = 1$

at $\mu = 1$: $\boxed{C = \begin{pmatrix} 5 \\ 9 \\ -1 \end{pmatrix}}$

b) $A = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} \quad B = \begin{pmatrix} -5 \\ 9 \\ -5 \end{pmatrix}$

$\overrightarrow{AC} = \begin{pmatrix} 5 \\ 9 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix} \rightarrow |\overrightarrow{AC}| = \sqrt{3^2+6^2+3^2}$
$= 3\sqrt{6}$

$\overrightarrow{BC} = \begin{pmatrix} 5 \\ 9 \\ -1 \end{pmatrix} - \begin{pmatrix} -5 \\ 9 \\ -5 \end{pmatrix} = \begin{pmatrix} 10 \\ 0 \\ 4 \end{pmatrix} \rightarrow |\overrightarrow{BC}| = \sqrt{4^2+10^2}$
$= 2\sqrt{29}$

$\overrightarrow{AC} \cdot \overrightarrow{BC} = \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 10 \\ 0 \\ 4 \end{pmatrix} = 42$

$\therefore \cos\theta = \dfrac{42}{3\sqrt{6} \times 2\sqrt{29}} = 0.531\ldots$

$\theta = \cos^{-1}(0.531..) = \boxed{57.95^\circ}$

c) Area $= \frac{1}{2} ab\sin C$

$= \frac{1}{2} \times 3\sqrt{6} \times 2\sqrt{29} \times \sin(57.95..) = \boxed{33.5}$ units^2

Q10a) $\underline{x = \sqrt{3}}$: $\sqrt{3} = \tan\theta$

$\theta = \tan^{-1}\sqrt{3} = \boxed{\frac{\pi}{3}}$

b) $x = \tan\theta$ $\quad\quad$ $y = \sin\theta$

$\frac{dx}{d\theta} = \sec^2\theta$ $\quad\quad$ $\frac{dy}{d\theta} = \cos\theta$

$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cos\theta}{\frac{1}{\cos^2\theta}} = \underline{\cos^3\theta}$

$\theta = \frac{\pi}{6}$, $\frac{dy}{dx} = \frac{1}{8}$ $\therefore$ at normal, $m = -8$

$(-8 \times \frac{1}{8} = -1)$

$\Rightarrow y - \frac{\sqrt{3}}{2} = -8(x - \sqrt{3})$

$\Rightarrow y = -8x + 8\sqrt{3} + \frac{\sqrt{3}}{2}$

$\Rightarrow y = -8x + \frac{17}{2}\sqrt{3}$

$\Rightarrow \underline{y = 0}$: $8x = \frac{17}{2}\sqrt{3}$ $\therefore x = \boxed{\frac{17}{16}\sqrt{3}}$

$\therefore Q\left(\frac{17}{16}\sqrt{3}, 0\right)$ $k = \frac{17}{16}$

c) $V = \pi \int_0^{\frac{\pi}{3}} \left[y^2 \frac{dx}{d\theta} \right] d\theta = \pi \int_0^{\pi/3} [\sin^2\theta \cdot \sec^2\theta] \, d\theta$

$= \pi \int_0^{\pi/3} [\tan^2\theta] \, d\theta = \pi \int_0^{\pi/3} [\sec^2\theta - 1] \, d\theta$

$(\tan^2\theta = \sec^2\theta - 1)$

$= \pi \left[\tan\theta - \theta \right]_0^{\frac{\pi}{3}} = \pi \left[\tan\frac{\pi}{3} - \frac{\pi}{3} \right]_0^{\frac{\pi}{3}}$

$= \pi \left[\sqrt{3} - \frac{\pi}{3} \right] - \pi [0] = \boxed{\sqrt{3}\pi - \frac{1}{3}\pi^2}$

Q11a) $\frac{dP}{dt} = \frac{1}{15}P(5-P)$

$$\frac{1}{P(5-P)}\frac{dP}{dt} = \frac{1}{15}$$

$$\int\left[\frac{1}{P(5-P)}\right]dP = \frac{1}{15}\int(1)\,dt$$

$\int\left[\frac{1}{P(5-P)}\right]dP$ ~ By Partial Fraction ...

$$\frac{1}{P(5-P)} = \frac{A}{P} + \frac{B}{5-P}$$

$$1 = A(5-P) + B(P)$$

$\underline{P=5}$: $1 = 5B$ $\therefore$ $B = \frac{1}{5}$ //

$\underline{P=1}$: $1 = 4A + \frac{1}{5}$ $\therefore$ $A = \frac{1}{5}$ //

$$\Rightarrow \int\left[\frac{\frac{1}{5}}{P} + \frac{\frac{1}{5}}{5-P}\right]dP = \frac{t}{15} + c$$

$$\Rightarrow \frac{1}{5}\ln P - \frac{1}{5}\ln|5-P| = \frac{t}{15} + c$$

$\underline{t=0, P=1}$: $-\frac{1}{5}\ln 4 = c$ //

$$\therefore \frac{1}{5}\ln\left|\frac{P}{5-P}\right| = \frac{t}{15} - \frac{1}{5}\ln 4$$

$$\frac{1}{5}\ln\left|\frac{P}{5-P}\right| + \frac{1}{5}\ln 4 = \frac{t}{15}$$

$$\frac{1}{5}\ln\left|\frac{4P}{5-P}\right| = \frac{t}{15}$$

$$\therefore t = 3\ln\left|\frac{4P}{5-P}\right|$$

$$\Rightarrow \frac{t}{3} = \ln\frac{4P}{5-P}$$

$$\Rightarrow e^{\frac{t}{3}} = \frac{4P}{5-P}$$

$$\Rightarrow (5-P)e^{\frac{t}{3}} = 4P$$

$$\Rightarrow 5e^{\frac{t}{3}} - Pe^{\frac{t}{3}} - 4P = 0$$

$$\Rightarrow P\left(4+e^{\frac{t}{3}}\right) = 5e^{\frac{t}{3}}$$

$$\Rightarrow P = \frac{(5e^{t/3})}{(4+e^{t/3})} \quad \begin{matrix} \leftarrow \div e^{t/3} \\ \leftarrow \div e^{t/3} \end{matrix}$$

$$\Rightarrow \boxed{P = \frac{5}{4e^{-t/3}+1}}$$

b) as $t \to \infty$, $P \to \frac{5}{1}$

$$P \to 5$$

hence P cannot exceed 5000.