C34 June 2018 (MA)

Q1i) $\int \frac{2x^2+5x+1}{x^2}\,dx$

$= \int \left[2 + \frac{5}{x} + x^{-2}\right] dx$

$= \boxed{\left[2x + 5\ln|x| - \frac{1}{x}\right] + c}$

ii) $\int [x\cos 2x]\,dx$ ~ By Parts

$\frac{dv}{dx} = \cos 2x \rightarrow v = \frac{1}{2}\sin 2x$

$u = x \rightarrow u' = 1$

$= \left[\frac{1}{2}x\sin 2x\right] - \frac{1}{2}\int (\sin 2x)\,dx$

$= \boxed{\frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + c}$

Q2a) $x = \frac{3}{2}t - 5 \longrightarrow \frac{dx}{dt} = \frac{3}{2}$

$y = 4 - 6t^{-1} \longrightarrow \frac{dy}{dt} = 6t^{-2}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{6}{t^2}}{\frac{3}{2}} = \frac{4}{t^2} \text{ //}$$

so at $t = 3$, $\frac{dy}{dx} = \frac{4}{3^2} = \boxed{\frac{4}{9}}$

b) $x = \frac{3}{2}t - 5$

$(x+5) \times \frac{2}{3} = t = \frac{2(x+5)}{3}$ //

$\therefore \; y = 4 - \frac{6}{\frac{2(x+5)}{3}}$

$y = 4 - \frac{9}{x+5} = \frac{4(x+5) - 9}{x+5}$

$y = \frac{4x + 20 - 9}{x+5}$

$\boxed{y = \frac{4x + 11}{x+5}}$ $\boxed{x \neq -5}$

denominator can't be 0 as anything divided by 0 is undefined.

(Q3a) $f(x) = 2^{x-1} - 4 + 1.5x = 0$

$$\frac{2^x}{2} - 4 + \frac{3}{2}x = 0$$

$\times 2$: $2^x - 8 + 3x = 0$

$\therefore \quad 3x = 8 - 2^x$

$\div 3 \Rightarrow \quad x = \frac{1}{3}(8 - 2^x)$

b) $x_1 = \frac{1}{3}(8 - 2^{1.6}) = \boxed{1.656}$

Similarly, $x_2 = \boxed{1.616}$

$x_3 = \boxed{1.645}$

c) $f(1.6325) = -0.0010\ldots$

$f(1.6335) = +0.00157\ldots$

change in sign between $x = 1.6325$ and $x = 1.6335$ indicates a root α lies between these values.

Hence $\alpha = 1.633$ to 3 d p

Q4a) $(1+px)^{-4} \simeq 1 - 4px + \frac{-4(-5)}{2!}(px)^2 + \frac{-4(-5)(-6)}{3!}(px)^3$

$\begin{bmatrix} x = px \\ n = -4 \end{bmatrix} \quad \simeq \boxed{1 - 4px + 10p^2x^2 - 20p^3x^3}$

b) $f(x) \simeq (3+4x)(1-4px+10p^2x^2-20p^3x^3)$

$\simeq 3 - 12px + 30p^2x^2 - 60p^3x^3 + 4x - 16px^2$
$+ 40p^2x^3 - 80p^3x^4$

$\simeq 3 + (4-12p)x + (30p^2-16p)x^2 + (40p^2-60p^3) \times x^3$
$+ \dots$

coeff of x^2 is twice that of x ...

so : $30p^2 - 16p = 2(4-12p)$

$\Rightarrow 30p^2 - 16p = 8 - 24p$

$\Rightarrow 30p^2 + 8p - 8 = 0$

$\Rightarrow 15p^2 + 4p - 4 = 0$

$\Rightarrow$ By Quadratic formula : $p = \frac{2}{5}$

$p = -\frac{2}{3}$

$p > 0 \quad \therefore \quad p = \boxed{\frac{2}{5}}$

c) from (b), coeff of $x^3 = (40p^2 - 60p^3)$

$= \left(40\left(\frac{2}{5}\right)^2 - 60\left(\frac{2}{5}\right)^3\right)$

$= \boxed{\frac{64}{25}}$

Q5 i a) $y = e^{2x} - 5$

$x \leftrightarrow y$; $x = e^{2y} - 5$

$x + 5 = e^{2y}$

$\ln(x+5) = 2y$

$\therefore y = \boxed{\frac{1}{2}\ln(x+5)} = f^{-1}(x)$.

domain : $\boxed{x > -5}$

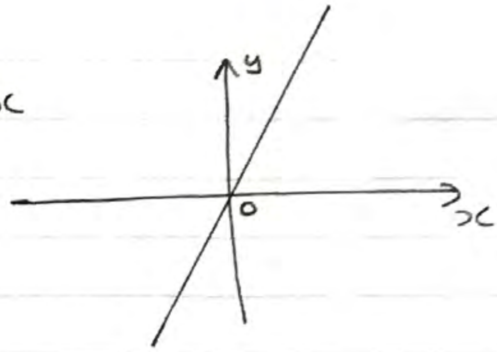
(as the function $y = \ln x$ is only defined for $x > 0$)

b) $g(3) = \ln(3(3) - 1) = \ln 8$

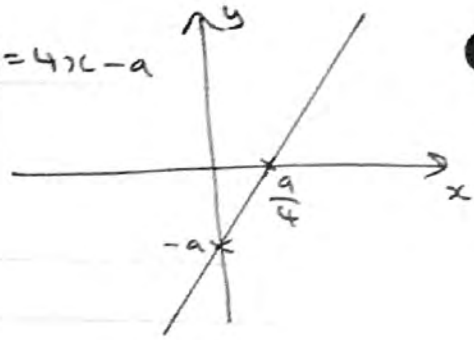
$fg(3) = f(\ln 8) = e^{2\ln 8} - 5 = e^{\ln 64} - 5$

$= 64 - 5 = \boxed{59}$

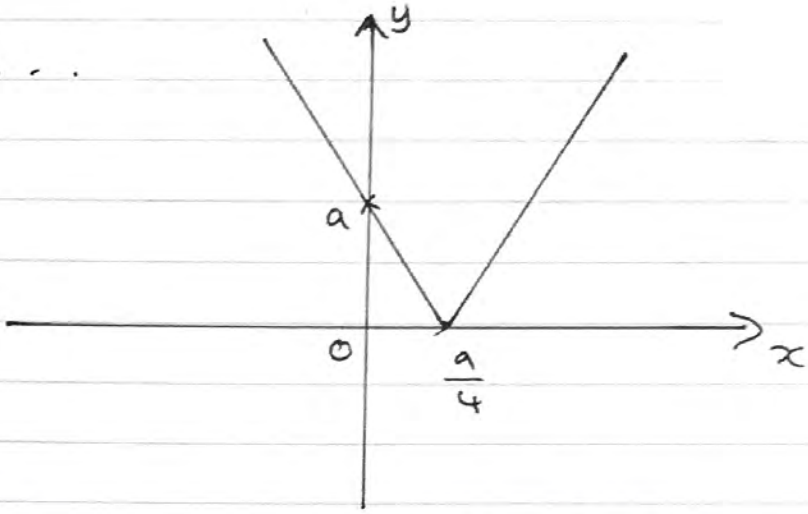
ii a) $y = 4x$



$y = 4x - a$



So $y = |4x - a|$ . . .



FIRST SOLUTION	2ND SOLUTION
b) $+(4x - a) = 9a$	$-(4x - a) = 9a$
$4x = 10a$	$a - 4x = 9a$
$x = \frac{10a}{4} = \frac{5a}{2}$ //	$8a = -4x$
	$\therefore x = -2a$ //

So our possible x-values are $\frac{5a}{2}$ and $-2a$.

$\underline{x = \frac{5}{2}a}$: $|x - 6a| + 3|x| = \left|\frac{-7a}{2}\right| + 3\left|\frac{5a}{2}\right|$

$= \boxed{11a}$

$\underline{x = -2a}$: $|x - 6a| + 3|x| = |-8a| + 3|-2a|$

$= \boxed{14a}$

(Q6a) $\sqrt{5}\cos\theta - 2\sin\theta \equiv R\cos(\theta+\alpha) \equiv R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

Comparing coefficients: $\sqrt{5} = R\cos\alpha$ ~ (1)

$2 = R\sin\alpha$ — (2)

$\frac{(2)}{(1)}$: $\frac{R\sin\alpha}{R\cos\alpha} = \frac{2}{\sqrt{5}} = \tan\alpha$

$\therefore \alpha = \tan^{-1}\left(\frac{2}{\sqrt{5}}\right) = \boxed{0.7297^c}$

Finding R: $R = \sqrt{(\sqrt{5})^2+(-2)^2} = \boxed{3}$

$\therefore \sqrt{5}\cos\theta - 2\sin\theta \equiv 3\cos(\theta + 0.7297)$

b) $\sqrt{5}\cos\theta - 2\sin\theta = \frac{1}{2}$

$3\cos(\theta + 0.7297) = \frac{1}{2}$

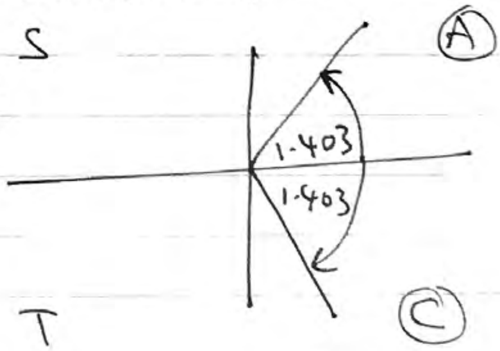
$\cos(\theta + 0.7297) = \frac{1}{6}$

$\therefore \theta + 0.7297 = \cos^{-1}\left(\frac{1}{6}\right) = 1.4033\ldots$

RANGE: $-2.412 < \theta + 0.7297 < 3.871$

$\theta + 0.7297 = -1.403, 1.403$

$\boxed{\theta = -2.13^c, 0.674^c}$



c) $f(x) = A\left(3\cos(\theta + 0.7297)\right) + B$.

so minimum is -15 for $f(x)$.
this will occur when $\cos(\theta + 0.7297) = -1$

$\Rightarrow -15 = A(3)(-1) + B$

$\Rightarrow -15 = -3A + B \quad \sim (1)$

and the maximum value of $f(x)$ is 33.
this will occur when $\cos(\theta + 0.7297) = 1$.

$\Rightarrow 33 = A(3)(1) + B$

$\Rightarrow 33 = 3A + B \quad \sim (2)$

$\underline{(1) + (2)}$: $-15 + 33 = 3A - 3A + 2B$

$\Rightarrow 2B = 18$
$\therefore \boxed{B = 9}$

and $3A + B = 33$

$\therefore A = \dfrac{33 - 9}{3} = \boxed{8}$

and $\boxed{A = -8}$

note that due to the symmetrical nature of the function $f(x)$, A can be 8 or -8 and the range of $f(x)$ will remain the same. The graph will just be reflected in the x-axis.

Q7) we require $\frac{dh}{dt}$ ...

$$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$$

$$\frac{dV}{dt} = 180 \text{ (given)}$$

$$V = \frac{1}{3}\pi h^2(90) - \frac{1}{3}\pi h^3$$

$$\therefore \frac{dV}{dh} = 60\pi h - \pi h^2$$

$$\text{so } \frac{dh}{dV} = \frac{1}{60\pi h - \pi h^2}$$

$$\Rightarrow \frac{dh}{dt} = 180 \times \frac{1}{60\pi h - \pi h^2} = \frac{180}{60\pi h - \pi h^2}$$

$$\text{at } h = 15, \ \frac{dh}{dt} = \frac{180}{60\pi(15) - \pi(15)^2} = \boxed{0.085}$$

(Q8a) assume intersection

$$\begin{pmatrix} 1+\lambda \\ 2\lambda-3 \\ 3\lambda+2 \end{pmatrix} = \begin{pmatrix} 6+\mu \\ 4+\mu \\ 1-\mu \end{pmatrix} \begin{matrix} \sim 1 \\ \sim 2 \\ \sim 3 \end{matrix}$$

$1: \quad \lambda = 5+\mu$ //

$\hookrightarrow 2: \quad 10+2\mu - 3 = 4+\mu$

$\mu = -3$ //

so from $1: \quad \lambda = 5-3 = 2$ //

sub $\lambda = 2$ into $3: \quad 3(2)+2 = 1-\mu$

$\mu = 1-2-6 = -7$ //

so values of μ/λ are not consistent.

$\therefore$ l_1 and l_2 do not intersect.

b) $\overrightarrow{OP} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$ $\quad \overrightarrow{OQ} = \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} 5-1 \\ 3--3 \\ 2-2 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 0 \end{pmatrix}$ //

$\begin{pmatrix} 4 \\ 6 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 4+12 = 16$ //

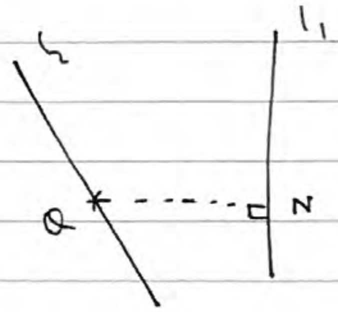
$$\left|\begin{pmatrix}4\\6\\0\end{pmatrix}\right| = \sqrt{4^2+6^2} = 2\sqrt{13}$$

$$\left|\begin{pmatrix}1\\2\\3\end{pmatrix}\right| = \sqrt{1^2+2^2+3^2} = \sqrt{14}\,.$$

$$\therefore \cos\theta = \frac{16}{2\sqrt{13}\times\sqrt{14}} = \frac{8}{\sqrt{182}}$$

$$\theta = \cos^{-1}\left(\frac{8}{\sqrt{182}}\right) = \boxed{53.63^\circ} \quad 2\text{ d.p.}$$

c) let N be closest point on l_1 to Q,



then $\overrightarrow{ON} = \begin{pmatrix}1+\lambda\\2\lambda-3\\3\lambda+2\end{pmatrix}$

$$\therefore \overrightarrow{QN} = \begin{pmatrix}1+\lambda\\2\lambda-3\\3\lambda+2\end{pmatrix} - \begin{pmatrix}5\\3\\2\end{pmatrix} = \begin{pmatrix}-4+\lambda\\2\lambda-6\\3\lambda\end{pmatrix} \quad //$$

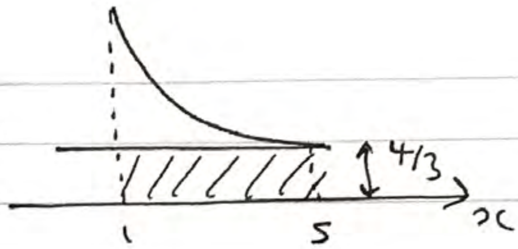
now since we're looking for the shortest distance, $\overrightarrow{QN}\cdot l_D = 0$ //

where l_{1D} represents the direction vector of l_1.

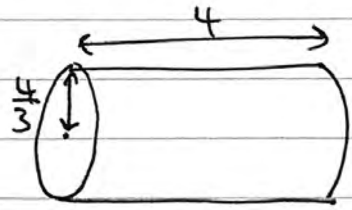
$$\Rightarrow \begin{pmatrix}\lambda-4\\2\lambda-6\\3\lambda\end{pmatrix}\cdot\begin{pmatrix}1\\2\\3\end{pmatrix} = 0$$

$$\Rightarrow \lambda-4+4\lambda-12+9\lambda=0$$

b) $V.R = \pi \int_1^5 [f(x)]^2 \, dx$ −



volume of shaded bit = volume of cylinder with radius $\frac{4}{3}$ and height 4.



$$\therefore Vol_{shaded} = \pi \left(\frac{4}{3}\right)^2 (4) = \boxed{\frac{64\pi}{9}}$$

$$\pi \int_1^5 [f(x)]^2 dx = 144\pi \int_1^5 \left[\frac{1}{(2x-1)^2}\right] dx$$

$$= 144\pi \left[-\frac{1}{2(2x-1)}\right]_1^5 = 144\pi\left[-\frac{1}{18}\right] - 144\pi\left[-\frac{1}{2}\right]$$

$$= 144\pi \left[\frac{1}{2} - \frac{1}{18}\right] = \frac{4}{9} \times 144\pi$$

$$= \boxed{64\pi}$$

So required volume $= 64\pi - \frac{64\pi}{9}$

$$= \boxed{\frac{512\pi}{9}}$$

Q10) $xe^{5-2y} - y = 0 \qquad P(2e^{-1}, 2)$

$$\frac{d}{dx}\left(xe^{5-2y} - y\right) = 0$$

$$e^{5-2y} + x(-2)e^{5-2y}\frac{dy}{dx} - \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\left(-2xe^{5-2y} - 1\right) = -e^{5-2y}$$

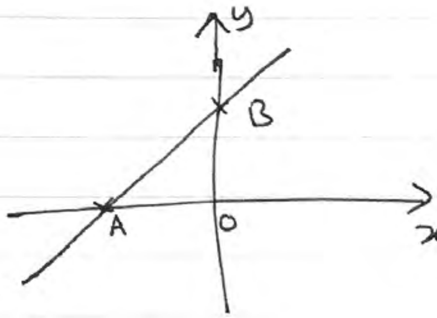
$$\therefore \frac{dy}{dx} = \frac{-e^{5-2y}}{-2xe^{5-2y} - 1} = \frac{e^{5-2y}}{1 + 2xe^{5-2y}} \quad //$$

at P, $\dfrac{dy}{dx} = \dfrac{e}{1 + \frac{4}{e}(e)} = \dfrac{e}{5} \quad //$

$$\therefore \quad y - 2 = \frac{e}{5}\left(x - 2e^{-1}\right) \quad \leftarrow \text{eqn of tangent at P.}$$

$$\Rightarrow y = \frac{xe}{5} - \frac{2}{5} + 2$$

$$\Rightarrow y = \frac{xe}{5} + \frac{8}{5} \quad //$$



$\underline{y = 0}$: $\quad \dfrac{xe}{5} + \dfrac{8}{5} = 0$

$$\therefore x = \frac{-8}{e} = (A) \quad //$$

$\underline{x = 0}$: $y = \frac{8}{5}$ //

$\therefore$ Area required $= \frac{1}{2} \times \frac{8}{e} \times \frac{8}{5} = \boxed{\frac{32}{5e}}$

Q11a) $x = \frac{3}{\cos\theta} = 3(\cos\theta)^{-1}$

$\therefore \frac{dx}{d\theta} = -3(\cos\theta)^{-2} \times -\sin\theta = \frac{3\sin\theta}{\cos^2\theta}$

$= 3\sin\theta\sec^2\theta$

$= \frac{3\sin\theta}{\cos\theta} \times \frac{1}{\cos\theta} = \underline{3\tan\theta\sec\theta}$

b) $R = \int_3^6 \frac{\sqrt{x^2-9}}{x}\,dx$

$x = 3\sec\theta$

$\frac{dx}{d\theta} = 3\sec\theta\tan\theta$.

$\therefore dx = 3\sec\theta\tan\theta\,d\theta$

x	θ
3	0
6	$\frac{\pi}{3}$

$= \int_0^{\pi/3} \frac{\sqrt{9\sec^2\theta - 9}}{\cancel{3}\cancel{\sec\theta}} \times \cancel{3}\cancel{\sec\theta}\tan\theta\,d\theta$

$= 3\int_0^{\pi/3} \frac{\sqrt{\sec^2\theta - 1}}{1} \times \tan\theta\,d\theta$

but $\sec^2\theta - 1 = \tan^2\theta$ //

$$\therefore \quad R = 3\int_0^{\pi/3} \left[\tan^2\theta\right] d\theta$$

$$= 3\int_0^{\pi/3} \left[\sec^2\theta - 1\right] d\theta$$

$$= 3\left[\tan\theta - \theta\right]_0^{\pi/3}$$

$$= 3\left[\tan\frac{\pi}{3} - \frac{\pi}{3}\right] - 3\left[0\right]$$

$$= 3\left[\sqrt{3} - \frac{\pi}{3}\right] = \boxed{3\sqrt{3} - \pi}$$

Q12a) $\text{LHS} = \cot x - \tan x$

$$= \frac{\cos x}{\sin x} - \frac{\sin x}{\cos x} = \frac{\cos^2 x}{\sin x\cos x} - \frac{\sin^2 x}{\sin x\cos x}$$

$$= \frac{\cos^2 x - \sin^2 x}{\sin x\cos x} = \frac{\cos 2x}{\frac{1}{2}\sin 2x}$$

$$= 2 \times \frac{\cos 2x}{\sin 2x} = \boxed{2\cot 2x}$$

(used x instead of θ)

b) $5 + 2\cot(2x - 30) = 0$

$2\cot(2x - 30) = -5$

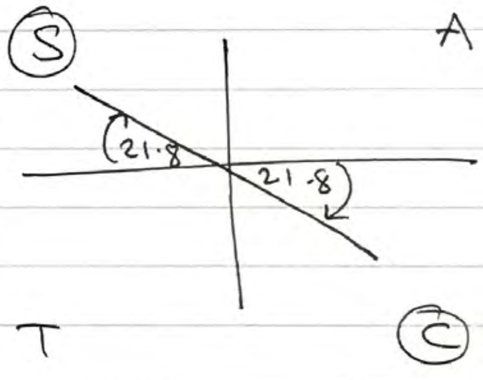
$\cot(2x - 30) = -\frac{5}{2}$

$\frac{1}{\tan(2x-30)} = -\frac{5}{2}$

$\therefore \tan(2x - 30) = -\frac{2}{5}$ //

so $\tan^{-1}\left(-\frac{2}{5}\right) = 2x - 30 = -21.8°$ //

RANGE : $-30 \le 2x - 30 < 330$



$2x - 30 = -21.8°, 158.2°$

$2x = 8.2°, 188.2°$

$\boxed{x = 4.1°, 94.1°}$

ie $\theta = 4.1°, 94.1°$

Q13a) $\dfrac{1}{(4-x)(2-x)} = \dfrac{A}{4-x} + \dfrac{B}{2-x}$

$1 = A(2-x) + B(4-x)$

$\underline{x=2}$: $1 = 4(B) - 2(B)$
$\therefore B = \frac{1}{2}$ //

$\underline{x=0}$: $1 = 2A + 2$
$\therefore A = -\frac{1}{2}$ //

so $\dfrac{1}{(4-x)(2-x)} = \dfrac{-\frac{1}{2}}{4-x} + \dfrac{\frac{1}{2}}{2-x}$ //

b) $\dfrac{dx}{dt} = k(4-x)(2-x)$

$\dfrac{1}{(4-x)(2-x)} \cdot \dfrac{dx}{dt} = k$

$\displaystyle\int \frac{1}{(4-x)(2-x)}\,dx = k\int (1)\,dt$

$\displaystyle\frac{1}{2}\int \frac{-1}{4-x} + \frac{1}{2-x}\,dx = kt + c$ //

$$\frac{1}{2}\left[\ln|4-x| - \ln|2-x|\right] = kt + c$$

$$\frac{1}{2}\ln\left|\frac{4-x}{2-x}\right| = kt + c$$

$\underline{t=0,\ x=0}$: $\frac{1}{2}\ln 2 = c \quad (= \ln\sqrt{2})$.

$$\therefore \quad \frac{1}{2}\ln\left|\frac{4-x}{2-x}\right| = kt + \frac{1}{2}\ln 2$$

$$kt = \frac{1}{2}\ln\left|\frac{4-x}{4-2x}\right|$$

$$2kt = \ln\left|\frac{4-x}{4-2x}\right|$$

$$\therefore \quad e^{2kt} = \frac{4-x}{4-2x}$$

$$4e^{2kt} - 2xe^{2kt} = 4-x$$

$$x - 2xe^{2kt} = 4 - 4e^{2kt}$$

$$x(1-2e^{2kt}) = 4-4e^{2kt}$$

$\div (1-2e^{2kt})$

$$\Rightarrow \quad x = \frac{4-4e^{2kt}}{1-2e^{2kt}}$$

c) from (b), $kt = \frac{1}{2}\ln\left|\frac{4-x}{4-2x}\right|$

$\Rightarrow \quad 0.1t = \frac{1}{2}\ln\left|\frac{4-1}{4-2}\right|$

$\Rightarrow \quad t = 5\ln\left|\frac{3}{2}\right| = \boxed{2.035}$

(Q14a) $y = \frac{(x^2-4)^{\frac{1}{2}}}{x^3}$ By QUOTIENT RULE

$u = (x^2-4)^{\frac{1}{2}} \rightarrow u' = \frac{1}{2}(x^2-4)^{-\frac{1}{2}}(2x)$

$v = x^3 \rightarrow v' = 3x^2$

$\frac{dy}{dx} = \frac{vu' - uv'}{v^2} = \frac{x^3\left(\frac{x}{\sqrt{x^2-4}}\right) - 3x^2\sqrt{x^2-4}}{x^6}$

$= \frac{\frac{x^4}{\sqrt{x^2-4}} - 3x^2\sqrt{x^2-4}}{x^6}$

× all terms by $\sqrt{x^2-4}$

$= \frac{x^4 - 3x^2(x^2-4)}{x^6\sqrt{x^2-4}}$

$$= \frac{x^4 - 3x^4 + 12x^2}{x^6\sqrt{x^2-4}}$$

$$= \frac{-2x^4 + 12x^2}{x^6\sqrt{x^2-4}} = \boxed{\frac{-2x^2+12}{x^4\sqrt{x^2-4}}}$$

$\div$ all terms by x^2.

$(A = -2)$

b) finding max point ; $\frac{dy}{dx} = 0$:

$$\frac{-2x^2+12}{x^4\sqrt{x^2-4}} = 0$$

$$-2x^2 + 12 = 0$$

$$x^2 = 6 \quad \therefore \quad x = \sqrt{6} \quad (x > 2)$$

at $x = \sqrt{6}$, $y = \frac{24\sqrt{6-4}}{(\sqrt{6})^3} = \frac{24\sqrt{2}}{6\sqrt{6}} = \frac{4}{\sqrt{3}}$

so $\boxed{0 < f(x) \leq \frac{4}{\sqrt{3}}}$

$y = 0$ is an asymptote.

c) The function $f(x)$ is a <u>many-to-one</u> function
f^{-1} would only exist if $f(x)$ was a one-to-one function