C34 June 17 IAL (MA)

Q1) $\frac{d}{dx}\left(3x^2 + 2xy - 2y^2 + 4 = 0\right)$

$$\Rightarrow 6x + 2y + 2x\frac{dy}{dx} - 4y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx}(2x - 4y) = -2y - 6x$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2(y+3x)}{2x-4y} = \frac{-\cancel{2}(y+3x)}{-\cancel{2}(2y-x)}$$

$$= \frac{3x+y}{2y-x}$$

at $(2,4)$, $m = \frac{3(2)+4}{2(4)-2} = \frac{5}{3}$

Substituting into eqn of a line...

$$y - 4 = \frac{5}{3}(x-2)$$

$$y = \frac{5}{3}x - \frac{10}{3} + 4$$

$$y = \frac{5}{3}x + \frac{2}{3}$$

$\times 3$: $3y = 5x + 2$

$$\boxed{5x - 3y + 2 = 0}$$

2) $\int_1^e [x^{-2}\ln x]\,dx$ $\sim$ $\frac{dv}{dx} = x^{-2}$

$v = -x^{-1}$

$$\boxed{\begin{aligned} u &= \ln x \\ u' &= \frac{1}{x} \end{aligned}}$$

$\Rightarrow \left[-\frac{\ln x}{x}\right]_1^e - \int_1^e \left[-\frac{1}{x} \times \frac{1}{x}\right] dx$

$\Rightarrow \left[-\frac{1}{e}\right] - [0] + \int_1^e (x^{-2})\,dx$

$\Rightarrow -\frac{1}{e} + \left[-\frac{1}{x}\right]_1^e = -\frac{1}{e} + \left[-\frac{1}{e}\right] - [-1]$

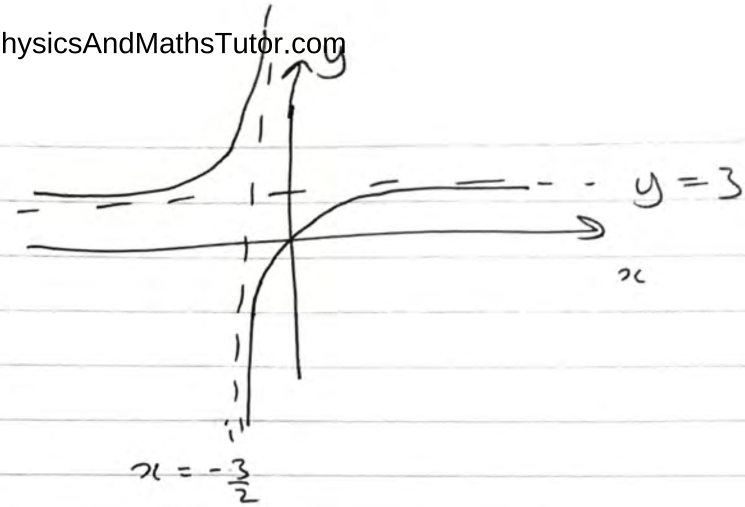
$= \boxed{1 - \frac{2}{e}}$

3a) $g(x) = \frac{6x}{2x+3}$

as $x \to \infty$, $g(x) \to 3$. $\therefore$ $y = 3$ is an asymptote.

$\therefore$ range: $\boxed{0 < g(x) < 3}$

rough sketch :



b) $y = \dfrac{6x}{2x+3}$

$x \leftrightarrow y$; $x = \dfrac{6y}{2y+3}$

make y the subject again,

$$2xy + 3x = 6y$$
$$6y - 2xy = 3x$$
$$y(6-2x) = 3x$$

$$y = \frac{3x}{6-2x} = g^{-1}(x) \qquad (0 < x < 3)$$

c) $g(x) = \dfrac{6x}{2x+3}$

$$\therefore gg(x) = \frac{6\left[\dfrac{6x}{2x+3}\right]}{2\left[\dfrac{6x}{2x+3}\right] + 3} = \frac{\dfrac{36x}{2x+3}}{\dfrac{12x}{2x+3} + 3}$$

$$= \frac{36x}{12x + 3(2x+3)} = \frac{36x}{18x+9}$$

$$= \boxed{\frac{4x}{2x+1}}$$

4a) $f(x) = 27(3-5x)^{-2}$

consider $(3-5x)^{-2}$,

$$(3-5x)^{-2} = 3^{-2}\left(1-\frac{5}{3}x\right)^{-2} = \frac{1}{9}\left(1-\frac{5}{3}x\right)^{-2}$$

$$\therefore (3-5x)^{-2} = \frac{1}{9}\left[1+\frac{10}{3}x+\frac{-2(-2-1)}{2}\left(-\frac{5}{3}x\right)^2+\frac{-2(-3)(-4)}{6}\left(\frac{-5}{3}x\right)^3\right.$$

$$\left[\begin{array}{l} n=-2 \\ x=-\frac{5}{3}x \end{array}\right]$$

$$= \frac{1}{9}\left[1+\frac{10}{3}x+\frac{25}{3}x^2+\frac{500}{27}x^3\right]+\ldots$$

$$= \frac{1}{9}+\frac{10}{27}x+\frac{25}{27}x^2+\frac{500}{243}x^3+\ldots$$

$$\therefore f(x) = 27\left[\frac{1}{9}+\frac{10}{27}x+\ldots\right]$$

$$= 3+10x+25x^2+\frac{500}{9}x^3+\ldots$$

b) sign change in 2^{nd} and 4^{th} term :

$$3 - 10x + 25x^2 - \frac{500}{9}x^3$$

c) 2^{nd} term will be $\div 5$ (5)

3^{rd} term will be $\div 5^2$ (25)

4^{th} term will be $\div 5^3$ (125)

$$\therefore 27(3-x)^{-2} = 3 + 2x + x^2 + \frac{4}{9}x^3$$

5a) $6 - 5x - 4x^2 = A(2-x)(1+2x) + B(1+2x) + C(2-x)$

let $x = 2$: $-20 = 5B$ $\therefore \boxed{B = -4}$

let $x = -\frac{1}{2}$: $\frac{15}{2} = \frac{5}{2}(C)$ $\therefore \boxed{C = 3}$

let $x = 0$: $6 = A(2) + B + 2C$

$A = \frac{6+4-6}{2} = \boxed{2 = A}$

b) $f(x) = 2 - 4(2-x)^{-1} + 3(1+2x)^{-1}$

$\therefore f'(x) = 4(2-x)^{-2}(-1) - 3(1+2x)^{-2}(2)$

$$= \frac{-4}{(2-x)^2} - \frac{6}{(1+2x)^2}$$

c) $f'(x) = \frac{-4}{(2-x)^2} - \frac{6}{(1+2x)^2}$

$(2-x)^2$ and $(1+2x)^2$ are always > 0
(for given domain).

so $f'(x) < 0$ always, $\therefore f(x)$ is a
decreasing function.

6) $l_1: \quad r = \begin{pmatrix} 5+6\lambda \\ 3\lambda-2 \\ 4-\lambda \end{pmatrix}$

$l_2: \quad r = \begin{pmatrix} 10+3\mu \\ 5+\mu \\ 2\mu-3 \end{pmatrix}$

assume intersection: $\begin{pmatrix} 5+6\lambda \\ 3\lambda-2 \\ 4-\lambda \end{pmatrix} = \begin{pmatrix} 10+3\mu \\ 5+\mu \\ 2\mu-3 \end{pmatrix} \begin{matrix} -1 \\ -2 \\ -3 \end{matrix}$

1 : $6\lambda - 3\mu = 5$

$\mu = \frac{6\lambda - 5}{3} = 2\lambda - \frac{5}{3}$ //

$\hookrightarrow$ 2 : $3\lambda - 2 = 5 + 2\lambda - \frac{5}{3}$

$\therefore \lambda = 16/3$ //

substitute 1 into 3 : $\lambda = 31/15$ //

$7 - \lambda = 4\lambda - \frac{10}{3}$

$5\lambda = \frac{31}{3}$

$\lambda = \frac{31}{15}$ //

Values of μ and λ are inconsistent $\therefore l_1$ and l_2 dont intersect.

direction vectors are : $\begin{pmatrix} 6 \\ 3 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$

$\begin{pmatrix} 6 \\ 3 \\ -1 \end{pmatrix} \neq a \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$ where a is any constant.

$\therefore$ they are not parallel.

Since they dont intersect and are not parallel we can conclude that l_1 and l_2 are $\boxed{\text{skew}}$

7a) $\text{LHS} = \dfrac{1-\cos 2x}{1+\cos 2x} = \dfrac{1-(2\cos^2 x - 1)}{1+2\cos^2 x - 1} = \dfrac{2-2\cos^2 x}{2\cos^2 x}$

$= \dfrac{\cancel{2}(1-\cos^2 x)}{\cancel{2}(\cos^2 x)} = \dfrac{\sin^2 x}{\cos^2 x} = \tan^2 x$ $\square$

b) $\dfrac{2(1-\cos 2\theta)}{(1+\cos 2\theta)} - 2 = 7\sec\theta$

$2\tan^2\theta - 2 = 7\sec\theta$

$1+\tan^2\theta = \sec^2\theta$

$2\sec^2\theta - 2 - 2 = 7\sec\theta$

$2\sec^2\theta - 7\sec\theta - 4 = 0$

$(2\sec\theta + 1)(\sec\theta - 4) = 0$

$\sec\theta = -\frac{1}{2}$

$\therefore \cos\theta = -2$ ✗

reject, no solutions

$\sec\theta = 4$.

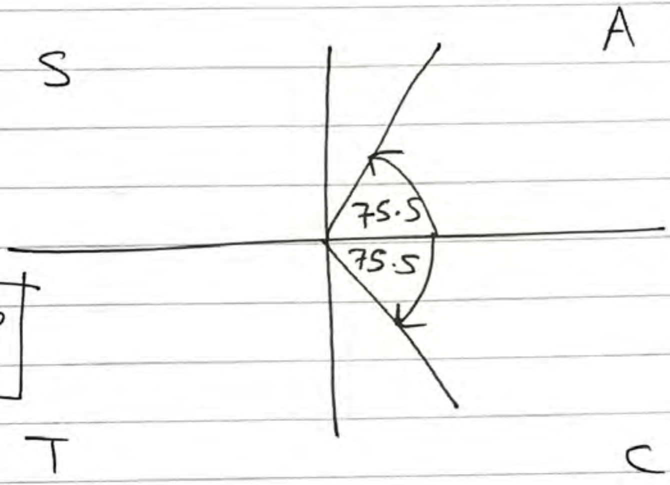
$\cos\theta = \frac{1}{4}$

$\therefore \theta = \cos^{-1}\left(\frac{1}{4}\right) = 75.5°$

By CAST

$-90 < \theta < 90°$

$\boxed{\theta = -75.5°, 75.5°}$



8a) Area $\simeq \frac{1}{2}(1)[0.6325 + 0.3742 + 2[0.5477 + 0.4851 + 0.4385 + 0.4027]]$

$h = \frac{b-a}{n} = \frac{7-2}{5} = 1$

$\therefore$ Area $\simeq \boxed{2.377}$

b) $V = \pi\int_2^7 y^2\,dx = \pi\int_2^7 \left(\frac{x}{x^2+1}\right)dx = \pi\left[\frac{1}{2}\ln[x^2+1]\right]_2^7$

By pattern.

$= \frac{1}{2}\pi[\ln 50] - \frac{1}{2}\pi[\ln 5] = \frac{1}{2}\pi[\ln 10]$

$= \boxed{\frac{\pi}{2}\ln 10}$

9a) $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$ $\quad \overrightarrow{AC} = \begin{pmatrix} 5 \\ -6 \\ 1 \end{pmatrix}$

$(\overrightarrow{AB}) \cdot (\overrightarrow{AC}) = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -6 \\ 1 \end{pmatrix} = -10$

$|\overrightarrow{AB}| \times |\overrightarrow{AC}| = \sqrt{17} \times \sqrt{62} = \sqrt{1054}$

$\therefore \cos(CAB) = \dfrac{-10}{\sqrt{1054}}$ $\quad \therefore \angle CAB \simeq \boxed{107.94^\circ}$

$(= -0.308)$

b) $|\overrightarrow{AB}| = \sqrt{17}$ $\quad$ angle $CAB = 107.94^\circ$

$|\overrightarrow{AC}| = \sqrt{62}$

$\therefore$ Area $= \frac{1}{2}ab\sin C = \frac{1}{2} \times \sqrt{17} \times \sqrt{62} \times \sin(107.94)$

$\simeq \boxed{15.44\,m^2}$

c) $\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC} = \begin{pmatrix} -2 \\ -3 \\ 2 \end{pmatrix} + \begin{pmatrix} 5 \\ -6 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \\ 3 \end{pmatrix}$ //

$\therefore |\overrightarrow{BC}| = \sqrt{3^2 + 9^2 + 3^2} = 3\sqrt{11}$.

Area $\triangle = \frac{1}{2}(BC) \times (\text{perp distance from } A \perp BC) = 15.44$ //

$\Rightarrow \frac{3}{2}\sqrt{11}\,(d) = 15.44$ // $\quad \therefore \boxed{d \simeq 3.10}$

10a) $2\sin\theta - \cos\theta \equiv R\sin(\theta - \alpha) \equiv R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$

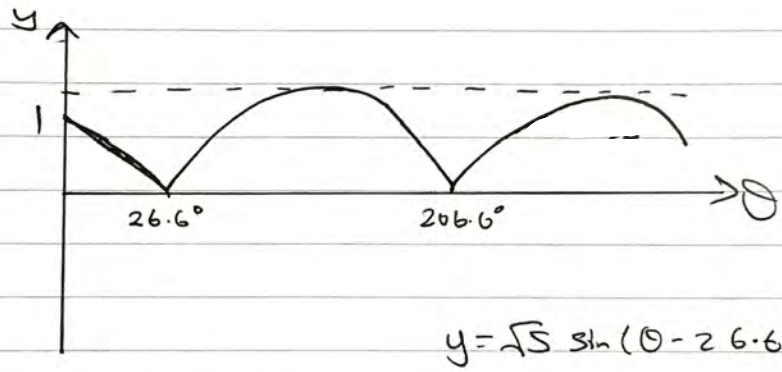
Comparing Coefficients: $2 = R\cos\alpha$ — ①
$1 = R\sin\alpha$ — ②

$\dfrac{②}{①}$: $\tan\alpha = \dfrac{1}{2}$ $\therefore \alpha = \tan^{-1}\left(\frac{1}{2}\right)$
$= \boxed{26.6^\circ}$

Finding R: $R = \sqrt{2^2 + (-1)^2} = \boxed{\sqrt{5}}$

$\therefore 2\sin\theta - \cos\theta \equiv \boxed{\sqrt{5}\sin(\theta - 26.6^\circ)}$

b)



$y = \sqrt{5}\sin(\theta - 26.6^\circ) = 0$

at y-axis, $y = \sqrt{5}\sin(\theta - 26.6^\circ) = -1$ $\therefore |-1| = 1$

at x-axis, $y = 0 = \sqrt{5}\sin(\theta - 26.6^\circ)$
$\therefore \sin(\theta - 26.6^\circ) = 0$
$\therefore \theta - 26.6^\circ = 0, 180$
$\theta = 26.6^\circ, 180 + 26.6^\circ$
$\theta = 26.6^\circ, 206.6^\circ$

i) $f(t) = 5 + \left| \sqrt{5} \sin(15t - 26.6^\circ) \right|$

$f(t)$ is max when $\sin(15t - 26.6^\circ) = 1$.

$\therefore [f(t)]_{max} = \boxed{5 + \sqrt{5}}$

ii) original function had range $0 \le \theta < 360^\circ$.

final maximum occurs at $\theta = 270^\circ$.

$\therefore 15t - 26.6^\circ = 270^\circ$

$15t = 296.6^\circ$

$\boxed{t = 19.8}$

11a) $y = 0 \rightarrow 2x^2 - 3 = 0$

$x^2 = \frac{3}{2} \quad \therefore x = \boxed{\sqrt{\frac{3}{2}}}$ //

($\tan\frac{x}{2}$ cannot be 0 in the given domain)

b) $\frac{dy}{dx} = 4x\tan(\frac{1}{2}x) + (2x^2-3)(\frac{1}{2}\sec^2(\frac{x}{2}))$ // $= 0$ at $x = \alpha$.

$\therefore 4\alpha\tan(\frac{\alpha}{2}) + \frac{1}{2}(2\alpha^2 - 3)\sec^2(\frac{\alpha}{2}) = 0$ //.

$$4\alpha \tan\left(\frac{\alpha}{2}\right) + \frac{1}{2}(2\alpha^2 - 3)\sec^2\left(\frac{\alpha}{2}\right) = 0$$

$\times \cos^2 \frac{\alpha}{2}$: $4\alpha \sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right) + \alpha^2 - \frac{3}{2} = 0.$

$$2\alpha\left(2\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)\right) + \alpha^2 - \frac{3}{2} = 0$$

$\Rightarrow \quad 2\alpha(\sin\alpha) + \alpha^2 - \frac{3}{2} = 0$

$\times 2$: $\quad 4\alpha\sin\alpha + 2\alpha^2 - 3 = 0.$

c) $x_1 = 0.7$

$x_2 = \dfrac{3}{2(0.7) + 4\sin(0.7)} = 0.7544$

$x_3 = 0.7062$ similarly.

d) considering $2\alpha^2 - 3 + (4\sin\alpha)\alpha = 0$

$\alpha = 0.72825$: $2(0.72825)^2 - 3 + 4(0.72825)\sin(0.72825)$

$= -0.0005\ldots < 0$

$\alpha = 0.72835$: $2(0.72835)^2 - 3 + 4(0.72835)\sin(0.72835)$

$= 0.00026\ldots > 0$

change in sign in the interval

$[0.72825, 0.72835]$

$\therefore$ a root α lies in this interval

$\therefore \alpha = 0.7283$ to 4dp

12a) $\frac{dV}{dt} = 0.4\pi - 0.2\pi\sqrt{h}$

$$\boxed{\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}}$$

$V = \pi r^2 h$

$\frac{dV}{dh} = \pi r^2$

$\therefore \frac{dh}{dV} = \frac{1}{\pi r^2} = \frac{1}{4\pi}$

$$\therefore \frac{dh}{dt} = \frac{0.4\pi - 0.2\pi\sqrt{h}}{4\pi}$$

$$\frac{dh}{dt} = \frac{0.4 - 0.2\sqrt{h}}{4}$$

$\times 20$: $20\frac{dh}{dt} = 2 - \sqrt{h}$

$\square$

b) $\left(\frac{20}{2 - h^{\frac{1}{2}}}\right)\frac{dh}{dt} = 1$

$$\therefore \int \frac{20}{2-\sqrt{h}}\,dh = \int (1)\,dt$$

without limits

hence time taken to get to $h = 2.25$ will be given by

$$\int_{0.16}^{2.25} \frac{20}{2-\sqrt{h}}\,dh = T.$$

c) $20\int_{0.16}^{2.25}\left[\frac{1}{2-\sqrt{h}}\right]dh$

$h = (2-x)^2$

$\frac{dh}{dx} = 2(2-x)(-1)$

$= -2(2-x)$

$\therefore dh = -2(2-x)\,dx$

h	x
0.16	1.6
2.25	0.5

$\boxed{\sqrt{h} = 2-x \quad x = 2-\sqrt{h}}$

$\Rightarrow 20\int_{1.6}^{0.5}\left[\frac{1}{2-(2-x)}\times -2(2-x)\right]dx$

$\Rightarrow -20\int_{1.6}^{0.5}\left[\frac{2(2-x)}{x}\right]dx$

$\Rightarrow (20)(2)\int_{0.5}^{1.6}\left[\frac{2}{x}-1\right]dx = 40\left[2\ln|x| - x\right]_{0.5}^{1.6}$

$= 40\left[2\ln 1.6 - 1.6\right] - 40\left[2\ln\frac{1}{2} - \frac{1}{2}\right]$

$= 40\left[2\ln(3.2) - 1.1\right] = 49.052$ min

$= \boxed{49 \text{ min}}$

13a) $t=0$: $P = 200 - \frac{160}{15+1} = 200 - \frac{160}{16}$

$= \boxed{190}$

$\therefore$ number of ants $= \boxed{190000}$.

b) $P = 200 - 160e^{0.6t}(15+e^{0.8t})^{-1}$.

By Product Rule:

$$\frac{dP}{dt} = -96e^{0.6t}(15+e^{0.8t})^{-1} + 160e^{0.6t}(15+e^{0.8t})^{-2}(0.8e^{0.8t})$$

$$\therefore \frac{dP}{dt} = \frac{-96e^{0.6t}}{15+e^{0.8t}} + \frac{128e^{1.4t}}{(15+e^{0.8t})^2}$$

$$= \frac{-96e^{0.6t}(15+e^{0.8t})+128e^{1.4t}}{(15+e^{0.8t})^2}$$

$$= \boxed{\frac{-1440e^{0.6t} + 32e^{1.4t}}{(15+e^{0.8t})^2}}$$

c) at T there is a minimum point.

setting $\frac{dP}{dt} = 0$:

$$-1440e^{0.6t} + 32e^{1.4t} = 0.$$

$$e^{0.6t}\left(-1440 + 32e^{0.8t}\right) = 0$$

$$e^{0.6t} \neq 0$$

$$\therefore \quad 32e^{0.8t} - 1440 = 0$$

$$e^{0.8t} = 45$$

$$0.8t = \ln 45$$

$$\therefore t = \frac{5}{4}\ln 45 \simeq \boxed{4.76}$$

14a) $\theta = \frac{\pi}{3}$: $x = 8\left(\cos\frac{\pi}{3}\right)^3 = 8\left(\frac{1}{2}\right)^3 = 1$ //

$y = 6\left(\sin\frac{\pi}{3}\right)^2 = 6\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{9}{2}$ //

$\therefore \boxed{P\left(1, \frac{9}{2}\right)}.$

b) $\frac{dy}{d\theta} = 12\sin\theta\cos\theta$ //

$\frac{dx}{d\theta} = 24\cos^2\theta(-\sin\theta)$ //

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = -\frac{12\cancel{\sin\theta}\cancel{\cos\theta}}{24\cos^{\cancel{2}}\theta\cancel{\sin\theta}}$$

$$= -\frac{1}{2\cos\theta}$$ //

b) cont. $\therefore$ at $\theta = \frac{\pi}{3}$, $m = -\frac{1}{2\left(\frac{1}{2}\right)} = -1$

so at normal, $\underline{m = 1}$.

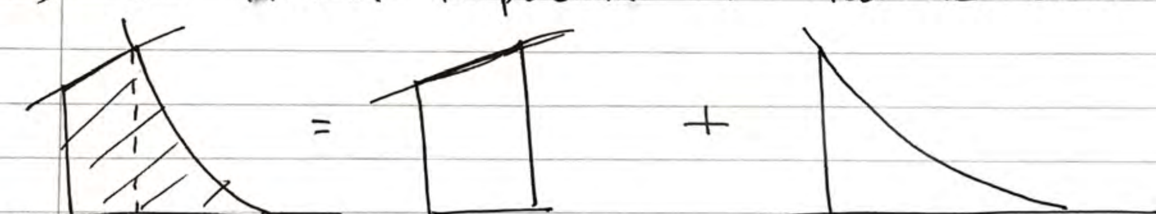
and $P\left(1, \frac{9}{2}\right)$.

$\therefore \quad y - \frac{9}{2} = 1\left(x - 1\right)$.

$y = x - 1 + \frac{9}{2}$

$\boxed{y = x + 3.5}$

c) S = Area of Trapezium + Area under curve



$$\text{Area}_{\text{TRAP}} = \frac{(a+b)h}{2} = \frac{(3.5+4.5)}{2} \times 1 = 4$$

$$\text{Area}_{\text{curve}} = \int_{\frac{\pi}{3}}^{0} y \frac{dx}{d\theta}\, d\theta = \int_{\frac{\pi}{3}}^{0} \left[6\sin^2\theta\left(-24\cos^2\theta\sin\theta\right)\right] d\theta$$

$$= -144\int_{\frac{\pi}{3}}^{0} \sin^2\theta\left(\sin\theta\cos^2\theta\right) d\theta = 144\int_{0}^{\pi/3} (1-\cos^2\theta)(\sin\theta\cos^2\theta)\, d\theta$$

$$= 144 \int_0^{\pi/3} \left[\sin\theta\cos^2\theta - \sin\theta\cos^4\theta\right] d\theta$$

$$\therefore R = \underbrace{4}_{\text{Trapezium}} + \underbrace{144 \int_0^{\pi/3} (\sin\theta\cos^2\theta - \sin\theta\cos^4\theta)\, d\theta}_{\text{area under curve}}.$$

d) evaluating the latter expression above,

$$144 \int_0^{\pi/3} \left[\sin\theta(\cos\theta)^2\right] d\theta \quad -144 \int_0^{\pi/3} \left[\sin\theta(\cos\theta)^4\right] d\theta$$

$$\Rightarrow \left[-\frac{144\cos^3\theta}{3}\right]_0^{\pi/3} + \left[\frac{144\cos^5\theta}{5}\right]_0^{\pi/3}$$

$$\Rightarrow \left[\frac{-144}{24}\right] + \left[\frac{144}{3}\right] + \left[\frac{9}{10}\right] - \left[\frac{144}{5}\right]$$

$$\Rightarrow \frac{141}{10}$$

$$\therefore R = 4 + \frac{141}{10} = \boxed{\frac{181}{10}}$$