1. A curve has equation

$$4x^2 - y^2 + 2xy + 5 = 0$$

The points P and Q lie on the curve.

Given that $\frac{dy}{dx} = 2$ at P and at Q,

(a) use implicit differentiation to show that $y - 6x = 0$ at P and at Q.

(6)

(b) Hence find the coordinates of P and Q.

(3)

a) $$8x - 2y\frac{dy}{dx} + 2x\frac{dy}{dx} + 2y = 0$$

$$\frac{dy}{dx}(2x - 2y) = -2y - 8x$$

$$\frac{dy}{dx} = \frac{-y - 4x}{x - y} = 2$$

$$- \quad - 4x = 2x - 2y$$

$$\boxed{y - 6x = 0}$$

b) $$y = 6x$$

$$4x^2 - 36x^2 + 2x(6x) + 5 = 0$$

$$20x^2 = 5$$

$$x^2 = \frac{1}{4} \rightarrow x = \pm\frac{1}{2}$$

$$\therefore \quad P\left(-\frac{1}{2}, -3\right)$$

$$Q\left(\frac{1}{2}, 3\right)$$

2. Given that

$$\frac{4(x^2+6)}{(1-2x)(2+x)^2} \equiv \frac{A}{(1-2x)} + \frac{B}{(2+x)} + \frac{C}{(2+x)^2}$$

(a) find the values of the constants A and C and show that $B = 0$

(4)

(b) Hence, or otherwise, find the series expansion of

$$\frac{4(x^2+6)}{(1-2x)(2+x)^2}, \quad |x| < \frac{1}{2}$$

in ascending powers of x, up to and including the term in x^2, simplifying each term.

(5)

a) $4(x^2+6) = A(2+x)^2 + B(1-2x)(2+x) + C(1-2x)$

let $x = \frac{1}{2}$

$25 = \frac{25}{4}A \longrightarrow \boxed{A=4}$

let $x = -2$

$40 = 5C \longrightarrow \boxed{C=8}$

Coefficient of x^2: $4 = A - 2B$

$4 = 4 - 2B \longrightarrow \boxed{B=0}$

b) $4(1-2x)^{-1} = 4\left[1 + 2x + \frac{(-1)(-2)}{2!}(-2x)^2\right]$

$= 4 + 8x + 16x^2$

$8(2+x)^{-2} = 8(2^{-2})\left(1+\frac{x}{2}\right)^{-2}$

$= 2\left[1 - x + \frac{(-2)(-3)}{2!}\left(\frac{x}{2}\right)^2\right]$

$= 2 - 2x + \frac{3}{2}x^2$

$4(1-2x)^{-1} + 8(2+x)^{-2} = 6 + 6x + \frac{35}{2}x^2$

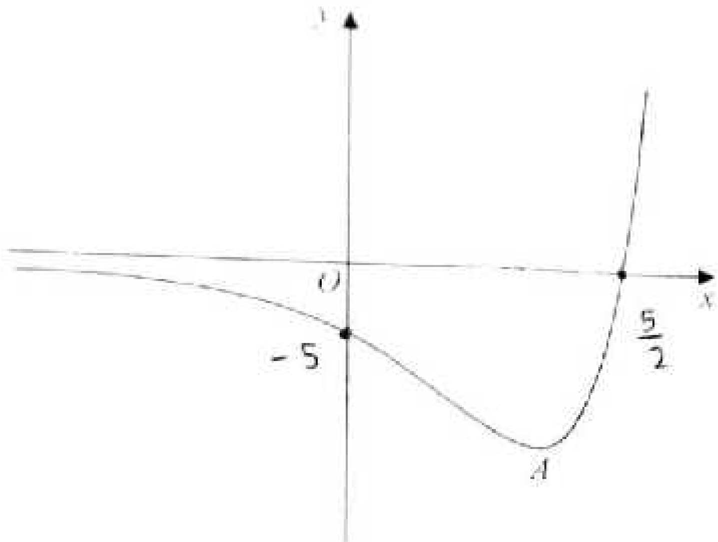


Figure 1

Figure 1 shows a sketch of part of the curve with equation $y = f(x)$, where

$$f(x) = (2x - 5)e^x, \qquad x \in \mathbb{R}$$

The curve has a minimum turning point at A.

(a) Use calculus to find the exact coordinates of A.

(5)

Given that the equation $f(x) = k$, where k is a constant, has exactly two roots,

(b) state the range of possible values of k.

(2)

(c) Sketch the curve with equation $y = |f(x)|$.

Indicate clearly on your sketch the coordinates of the points at which the curve crosses or meets the axes.

(3)

a) $f'(x) = 2e^x + (2x-5)e^x = 0$

$e^x(-3+2x) = 0 \rightarrow x = \frac{3}{2}$

$y = (2(\frac{3}{2}) - 5)e^{3/2}$

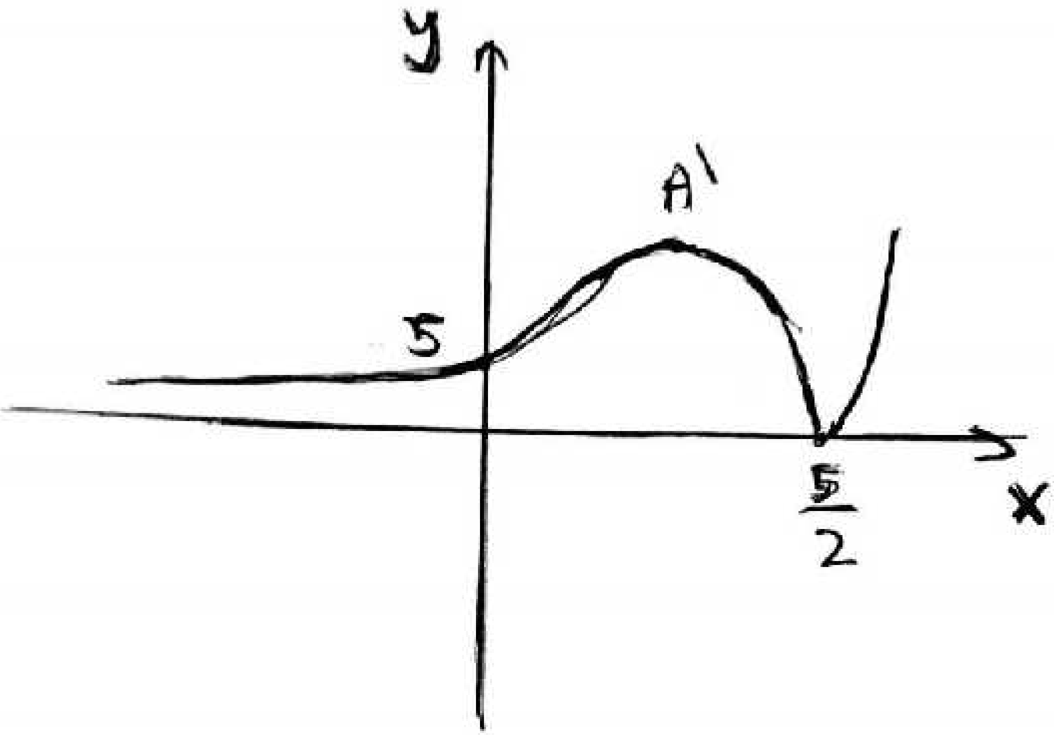
$= -2e^{3/2}$

$A(\frac{3}{2}, -2e^{3/2})$

b) $y = 0$ is a asymptote

$\therefore -2e^{3/2} < K < 0$

Question 3 continued



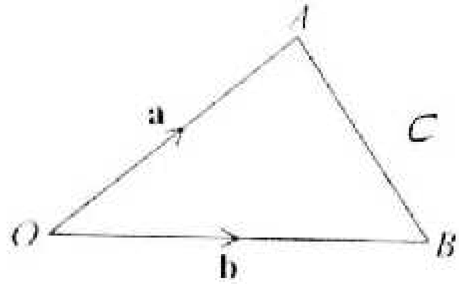


Figure 2

Figure 2 shows the points A and B with position vectors **a** and **b** respectively, relative to a fixed origin O.

Given that $|\mathbf{a}| = 5$, $|\mathbf{b}| = 6$ and $\mathbf{a}.\mathbf{b} = 20$

(a) find the cosine of angle AOB.

(2)

(b) find the exact length of AB.

(2)

(c) Show that the area of triangle OAB is $5\sqrt{5}$

(3)

a) $\cos AOB = \dfrac{a \cdot b}{|a|\,|b|} = \dfrac{20}{30} = \dfrac{2}{3}$

b) $c^2 = a^2 + b^2 - 2ab\cos AOB$

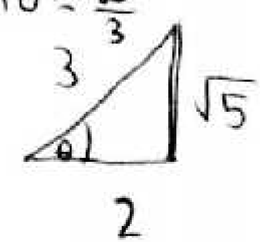
$= 25 + 36 - 60\left(\frac{2}{3}\right) = 21$

$AB = \sqrt{21}$

c) $\text{Area} = \frac{1}{2}ab\sin AOB$

$\cos\theta = \frac{2}{3}$

$\sin\theta = \frac{\sqrt{5}}{3}$



$= \frac{1}{2}(5)(6)\left(\frac{\sqrt{5}}{3}\right)$

$= 15\left(\frac{\sqrt{5}}{3}\right) = \boxed{5\sqrt{5}}$

5. (i) Find the x coordinate of each point on the curve $y = \frac{x}{x+1}$, $x \neq -1$, at which the gradient is $\frac{1}{4}$

(4)

(ii) Given that

$$\int_a^{2a} \frac{t+1}{t} \, dt = \ln 7 \qquad a > 0$$

find the exact value of the constant a.

(4)

i) $y' = \frac{(x+1) - x}{(x+1)^2} = \frac{1}{(x+1)^2} = \frac{1}{4}$

$4 = (x+1)^2$

$\pm 2 = x + 1$

$x = 1$ or $x = -3$

ii) $\int_a^{2a} \left(1 + \frac{1}{t}\right) dt = \left[t + \ln|t|\right]_a^{2a}$

$= 2a + \ln(2a) - a - \ln(a)$

$= a + \ln(2) + \cancel{\ln(a)} - \cancel{\ln(a)}$

$a + \ln 2 = \ln 7$

$\boxed{a = \ln\left(\frac{7}{2}\right)}$

6. The mass, m grams, of a radioactive substance t years after first being observed, is modelled by the equation

$$m = 25e^{1-kt}$$

where k is a positive constant.

(a) State the value of m when the radioactive substance was first observed. **(1)**

Given that the mass is 50 grams, 10 years after first being observed,

(b) show that $k = \frac{1}{10}\ln\left(\frac{1}{2}e\right)$ **(4)**

(c) Find the value of t when $m = 20$, giving your answer to the nearest year. **(3)**

a) $t = 0 \rightarrow m = 25e$

b)
$$50 = 25e^{1-10k}$$
$$e^{1-10k} = 2$$
$$1 - 10k = \ln(2)$$
$$-10k = \ln(2) - 1$$
$$k = \frac{1}{10} - \frac{1}{10}\ln(2)$$
$$= \frac{1}{10}\ln e - \frac{1}{10}\ln(2)$$
$$= \frac{1}{10}\left[\ln e - \ln 2\right] = \frac{1}{10}\ln\frac{e}{2}$$

c)
$$\frac{20}{25} = e^{1-kt}$$
$$\ln\left(\frac{4}{5}\right) = 1 - kt$$
$$kt = 1 - \ln\left(\frac{4}{5}\right) \Rightarrow t = \frac{1}{k}\left[1 - \ln\left(\frac{4}{5}\right)\right]$$
$$t = 39.86\ldots$$

7. (a) Use the substitution $t = \tan x$ to show that the equation

$$4\tan 2x - 3\cot x \sec^2 x = 0$$

can be written in the form

$$3t^4 + 8t^2 - 3 = 0$$

(4)

(b) Hence solve, for $0 \leqslant x < 2\pi$,

$$4\tan 2x - 3\cot x \sec^2 x = 0$$

Give each answer in terms of π. You must make your method clear.

(4)

a) $4\left(\frac{2\tan x}{1-\tan^2 x}\right) - 3\cot x\left(1+\tan^2 x\right) = 0$

$= \frac{8\tan x}{1-\tan^2 x} - 3\frac{1}{\tan x} - 3\tan x = 0$

let $t = \tan x$

$\frac{8t}{1-t^2} - \frac{3}{t} - 3t = 0 \quad * \, t(1-t^2)$

$8t^2 - 3(1-t^2) - 3t^2(1-t^2) = 0$

$8t^2 - 3 + \cancel{3t^2} - \cancel{3t^2} + 3t^4 = 0$

$3t^4 + 8t^2 - 3 = 0$

b)

$(3t^2 - 1)(t^2 + 3) = 0$

$t^2 = \frac{1}{3} \qquad t^2 = -3$

$\tan x = \frac{1}{\sqrt{3}}$, $\tan x = -\frac{1}{\sqrt{3}}$ rejected

$x = \boxed{\frac{1}{6}\pi} \qquad x = -\frac{1}{6}\pi \qquad -\frac{1}{6}\pi + 2\pi = \boxed{\frac{11}{6}\pi}$

$\pi + \frac{1}{6}\pi = \boxed{\frac{7}{6}\pi} \qquad \pi + \left(-\frac{1}{6}\pi\right) = \boxed{\frac{5}{6}\pi}$

$x = \frac{1}{6}\pi, \frac{5}{6}\pi, \frac{7}{6}\pi, \frac{11}{6}\pi$

8. (a) Prove by differentiation that

$$\frac{d}{dy}(\ln \tan 2y) = \frac{4}{\sin 4y}, \qquad 0 < y < \frac{\pi}{4}$$

(4)

(b) Given that $y = \frac{\pi}{6}$ when $x = 0$, solve the differential equation

$$\frac{dy}{dx} = 2\cos x \sin 4y, \qquad 0 < y < \frac{\pi}{4}$$

Give your answer in the form $\tan 2y = Ae^{B\sin x}$, where A and B are constants to be determined.

(6)

a) $\frac{d}{dy}(\ln \tan 2y) = \frac{2\sec^2 2y}{\tan 2y} = \frac{\frac{2}{\cos^2 2y}}{\frac{\sin 2y}{\cos 2y}}$

$= \frac{2}{\sin 2y \cos 2y} = \frac{4}{2\sin 2y \cos 2y} = \boxed{\frac{4}{\sin 4y}}$

b) $\int \frac{1}{\sin 4y}\, dy = \int 2\cos x \, dx$

$\frac{\ln(\tan 2y)}{4} = 2\sin x + C$

$x = 0,\ y = \frac{\pi}{6} \rightarrow \qquad \frac{\ln\sqrt{3}}{4} = 0 + C$

$C = \frac{1}{8}\ln(3)$

$\ln \tan 2y = 8\sin x + \frac{1}{2}\ln(3)$

$\tan 2y = e^{8\sin x + \ln\sqrt{3}}$

$= e^{8\sin x} e^{\ln\sqrt{3}}$

$\boxed{\tan 2y = \sqrt{3}\, e^{8\sin x}}$

9.

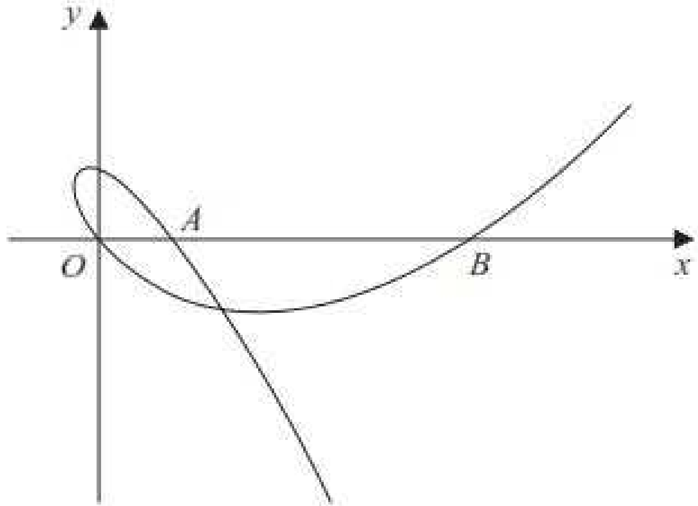


Figure 3

Figure 3 shows a sketch of part of the curve with parametric equations

$$x = t^2 + 2t, \qquad y = t^3 - 9t, \qquad t \in \mathbb{R}$$

The curve cuts the x-axis at the origin and at the points A and B as shown in Figure 3.

(a) Find the coordinates of point A and show that point B has coordinates $(15, 0)$.

(3)

(b) Show that the equation of the tangent to the curve at B is $9x - 4y - 135 = 0$

(5)

The tangent to the curve at B cuts the curve again at the point X.

(c) Find the coordinates of X.

(5)

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

a) $0 = t^3 - 9t$, $\quad t(t^2 - 9) = 0$

$t = 0, \quad t = \pm 3$

$t = -3 \rightarrow x = 9 - 6 = 3 \qquad t = 3 \rightarrow x = 15$

$A(3, 0) \qquad B(15, 0)$

b) $\dfrac{dx}{dt} = 2t + 2 \qquad \dfrac{dy}{dt} = 3t^2 - 9$

$\dfrac{dy}{dt} = \dfrac{3t^2 - 9}{2t + 2} \quad$ at $t = 3 \rightarrow y' = \dfrac{9}{4}$

$\dfrac{y - 0}{x - 15} = \dfrac{9}{4}$

c) $9(t^2+2t)-4(t^3-9t)-135=0$

$9t^2+18t-4t^3+36t-135=0$

$-4t^3+9t^2+54t-135=0$

$4t^3-9t^2-54t+135=0$

$f(t)=4t^3-9t^2-54t+135$

$f(3)=4(3)^3-9(3)^2-54(3)+135$

$=108-81-162+135$

$=27-162+135=0$

$\therefore$ $t-3$ is a factor

$$\begin{array}{r l}
 & 4t^2+3t-45 \\
t-3 & \overline{\big)\,4t^3-9t^2-54t+135} \\
\ominus & 4t^3-12t^2 \\
\hline
\ominus & 3t^2-54t \\
 & 3t^2-9t \\
\hline
\ominus & -45t+135 \\
 & -45t+135 \\
\hline
 & 0 \quad 0
\end{array}$$

$(t-3)(4t^2+3t-45)=0$

$(t-3)(4t+15)(t-3)=0$

$(4t+15)(t-3)^2=0$

$-4t=15$

$t=\frac{-15}{4}$

$x=\left(-\frac{15}{4}\right)^2+2\left(-\frac{15}{4}\right)=\frac{105}{16}$

$y=\left(-\frac{15}{4}\right)^3-9\left(-\frac{15}{4}\right)=-\frac{1215}{64}$

$X\left(\frac{105}{16},-\frac{1215}{64}\right)$

10.

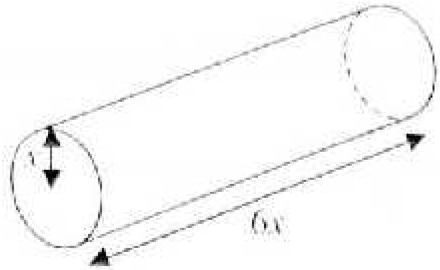


Figure 4

Figure 4 shows a right circular cylindrical rod which is expanding as it is heated.

At time t seconds the radius of the rod is x cm and the length of the rod is $6x$ cm.

Given that the **cross-sectional area** of the rod is increasing at a constant rate of $\frac{\pi}{20}$ cm^2 s^{-1}, find the rate of increase of the volume of the rod when $x = 2$

Write your answer in the form $k\pi$ cm^3 s^{-1} where k is a rational number.

(6)

$$\frac{dA}{dt} = \frac{\pi}{20}$$

$$\frac{dV}{dt} = \frac{dV}{dA} \times \frac{dA}{dt}$$

$$= \frac{9}{\sqrt{\pi}} \sqrt{A} \left(\frac{\pi}{20}\right)$$

$$= \frac{9}{20} \sqrt{A} \sqrt{\pi}$$

$x=2$

$$= \frac{9}{20} \sqrt{4\pi} \sqrt{\pi}$$

$$\frac{dV}{dt} = \boxed{\frac{9}{10}\pi}$$

$$V = 6x\,(\pi x^2)$$

$$A = \pi x^2$$

$$x = \sqrt{\frac{A}{\pi}}$$

$$V = 6x\,(A)$$

$$V = 6\left(\sqrt{\frac{A}{\pi}}\right)(A)$$

$$V = \frac{6}{\sqrt{\pi}} A^{3/2}$$

$$\frac{dV}{dA} = \frac{9}{\sqrt{\pi}} A^{1/2}$$

$$x = 2 \rightarrow \quad A = 4\pi$$

11. (a) Express $1.5\sin\theta - 1.2\cos\theta$ in the form $R\sin(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the value of R and the value of α to 3 decimal places.

(3)

The height, H metres, of sea water at the entrance to a harbour on a particular day, is modelled by the equation

$$H = 3 + 1.5\sin\left(\frac{\pi t}{6}\right) - 1.2\cos\left(\frac{\pi t}{6}\right), \quad 0 \leqslant t < 12$$

where t is the number of hours after midday.

(b) Using your answer to part (a), calculate the minimum value of H predicted by this model and the value of t, to 2 decimal places, when this minimum occurs.

(4)

(c) Find, to the nearest minute, the times when the height of sea water at the entrance to the harbour is predicted by this model to be 4 metres.

(6)

a) $1.5\sin\theta - 1.2\cos\theta = R\sin\theta\cos\alpha - R\sin\alpha\cos\theta$

$R\cos\alpha = 1.5$

$R\sin\alpha = 1.2$ $\qquad R = \sqrt{1.5^2 + 1.2^2} = \dfrac{3\sqrt{41}}{10}$

$\tan\alpha = \dfrac{4}{5}$

$= 1.921$

$\alpha = 0.675$

b) $H = 3 + \dfrac{3\sqrt{41}}{10}\sin\left(\dfrac{\pi}{6}t - 0.675\right)$, $H_{min.} = 3 - \dfrac{3\sqrt{41}}{10}$

$\dfrac{\pi}{6}t - 0.675 = \dfrac{3}{2}\pi$ $\qquad = 1.08$ metres

$\dfrac{\pi}{6}t = 5.387$ $\qquad \boxed{t = 10.29 \text{ hours}}$

c) $1 = \dfrac{3\sqrt{41}}{10}\sin\left(\dfrac{\pi}{6}t - 0.675\right)$

$\dfrac{\pi}{6}t - 0.675 = 0.5475$, $\quad \dfrac{\pi}{6}t - 0.675 = \pi - 0.5475$

$t = 2.33$ h $\qquad t = 6.2435$ h

$= 139.8$ min $\qquad = 374.61$ min.

$\sim \boxed{140 \text{ min.}}$ $\qquad \sim \boxed{375 \text{ min.}}$

12. (i) Relative to a fixed origin O, the line l_1 is given by the equation

$$l_1:\ \mathbf{r} = \begin{pmatrix} -5 \\ 1 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \text{ where } \lambda \text{ is a scalar parameter.}$$

The point P lies on l_1. Given that $\overrightarrow{OP}$ is perpendicular to l_1, calculate the coordinates of P.

(5)

(ii) Relative to a fixed origin O, the line l_2 is given by the equation

$$l_2:\ \mathbf{r} = \begin{pmatrix} 4 \\ -3 \\ 12 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ -3 \\ 4 \end{pmatrix} \text{ where } \mu \text{ is a scalar parameter.}$$

The point A **does not** lie on l_2. Given that the vector $\overrightarrow{OA}$ is parallel to the line l_2 and $|\overrightarrow{OA}| = \sqrt{2}$ units, calculate the possible position vectors of the point A.

(5)

i) $\overrightarrow{OP} = \begin{pmatrix} -5+2\lambda \\ 1-3\lambda \\ 6+\lambda \end{pmatrix}$, $\begin{pmatrix} -5+2\lambda \\ 1-3\lambda \\ 6+\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0$

$-10+4\lambda-3+9\lambda+6+\lambda=0$

$14\lambda = 7 \quad \boxed{\lambda = \frac{1}{2}}$

$P = \begin{pmatrix} -4 \\ -0.5 \\ 6.5 \end{pmatrix}$

ii) $\overrightarrow{OA} = \begin{pmatrix} 5t \\ -3t \\ 4t \end{pmatrix}$ $\quad (5t)^2+(-3t)^2+(4t)^2=(\sqrt{2})^2$

$50t^2=2$

$t^2=\frac{1}{25}$

$t = \pm\frac{1}{5}$

$\overrightarrow{OA} = \begin{pmatrix} -1 \\ \frac{3}{5} \\ -\frac{4}{5} \end{pmatrix}$ or $\begin{pmatrix} 1 \\ -\frac{3}{5} \\ \frac{4}{5} \end{pmatrix}$

13.

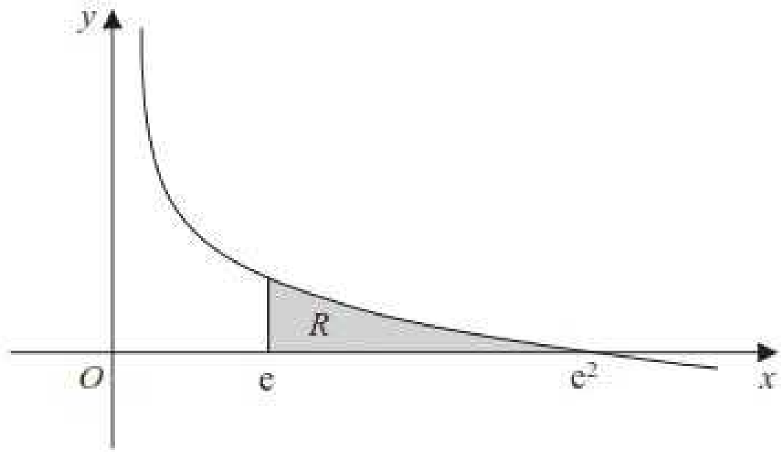


Figure 5

Figure 5 shows a sketch of part of the curve with equation $y = 2 - \ln x, \quad x > 0$

The finite region R, shown shaded in Figure 5, is bounded by the curve, the x-axis and the line with equation $x = e$.

The table below shows corresponding values of x and y for $y = 2 - \ln x$

x	e	$\frac{e + e^2}{2}$	e^2
y	1		0

(a) Complete the table giving the value of y to 4 decimal places.

(1)

(b) Use the trapezium rule, with all the values of y in the completed table, to obtain an estimate for the area of R, giving your answer to 3 decimal places.

(3)

(c) Use integration by parts to show that $\int (\ln x)^2 \, dx = x(\ln x)^2 - 2x\ln x + 2x + c$

(4)

The area R is rotated through 360° about the x-axis.

(d) Use calculus to find the exact volume of the solid generated.

Write your answer in the form $\pi e(pe + q)$, where p and q are integers to be found.

(6)

b) $h = \frac{e + e^2}{2} - e = \frac{e^2 - e}{2}$

$\frac{1}{2}\left(\frac{e^2 - e}{2}\right)\left[1 + 0 + 2(0.3799)\right] = 2.055$

c) $I = \int (\ln x)^2 \, dx$ $\quad u = (\ln x)^2 \quad dv = dx$

$du = \frac{2\ln x}{x} dx \quad v = x$

$I = x(\ln x)^2 - \int 2 \ln x \, dx$

$u = \ln x \quad dv = dx$

$du = \frac{1}{x} dx \quad v = x$

$I = x(\ln x)^2 - 2\left[x \ln x - \int dx \right]$

$= x(\ln x)^2 - 2x\ln x + 2x + c$

d) $V = \pi \int y^2 \, dx = \pi \int_e^{e^2} (2 - \ln x)^2 \, dx$

$= \pi \int_e^{e^2} (4 - 4\ln x + (\ln x)^2) \, dx$

$= \pi \left[4x - 4x\ln x + 4x + x(\ln x)^2 - 2x\ln x + 2x \right]_e^{e^2}$

$= \pi \left[10x - 6x\ln x + x(\ln x)^2 \right]_e^{e^2}$

$= \pi \left[10e^2 - 6e^2 \ln e^2 + e^2(\ln e^2)^2 - 10e + 6e\ln e - e(\ln e)^2 \right]$

$= \pi \left[2e^2 - 5e \right] = \boxed{\pi e [2e - 5]}$