

C34 January 2018 (MA)

$$(Q1) \frac{d}{dx} (3^x + xy = x + y^2)$$

$$3^x \ln 3 + y + x \frac{dy}{dx} = 1 + 2y \frac{dy}{dx}$$

$$\frac{dy}{dx} (x - 2y) = 1 - y - 3^x \ln 3$$

$$\therefore \frac{dy}{dx} = \frac{1 - y - 3^x \ln 3}{x - 2y}$$

$$\text{At } P(4, 11), \frac{dy}{dx} = \frac{1 - 11 - 3^4 \ln 3}{4 - 2(11)}$$

$$= \frac{-10 - 81 \ln 3}{-18}$$

$$= \boxed{\frac{5}{9} + \frac{9 \ln 3}{2}}$$

$$\begin{aligned} \text{Q2a)} \quad (125 - 5x)^{\frac{2}{3}} &= 125^{\frac{2}{3}} \left(1 - \frac{5}{125}x\right)^{\frac{2}{3}} \\ &= 25 \left(1 - \frac{x}{25}\right)^{\frac{2}{3}} \quad \left[\begin{array}{l} n = -\frac{x}{25} \\ n = \frac{2}{3} \end{array} \right] \end{aligned}$$

$$(125 - 5x)^{\frac{2}{3}} \approx 25 \left[1 - \frac{2}{75}x + \frac{-\frac{2}{9}}{2} \left(\frac{-x}{25}\right)^2 \right]$$

$$\approx 25 \left[1 - \frac{2}{75}x - \frac{x^2}{5625} \right]$$

$$\approx \boxed{25 - \frac{2}{3}x - \frac{x^2}{225}}$$

$$\text{b)} \quad 125 - 5x = 120.$$

$$5x = 5$$

$$x = 1$$

sub $x=1$ into answer from (a):

$$(120)^{\frac{2}{3}} \approx 25 - \frac{2}{3}(1) - \frac{(1)}{225}$$

$$\approx \boxed{24.32889}$$

$$\bullet \text{ (Q3a)} \quad \frac{x^2}{4} + \ln(2x) = 0$$

$$\ln(2x) = -\frac{x^2}{4}$$

$$2x = e^{-\frac{x^2}{4}}$$

$$\therefore x = \boxed{\frac{1}{2} e^{-\frac{x^2}{4}}}$$

$$\bullet \text{ b) } x_2 = \frac{1}{2} e^{-\frac{1}{4}(0.5)^2} = \boxed{0.4697}$$

$$\text{similarly, } \boxed{x_3 = 0.4732}$$

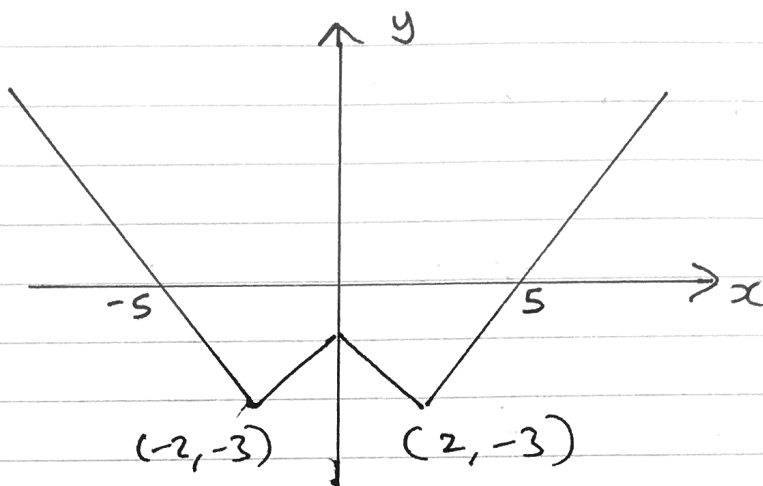
$$\boxed{x_4 = 0.4728}$$

$$\bullet \text{ c) } f(0.4725) = -0.000756 \dots$$

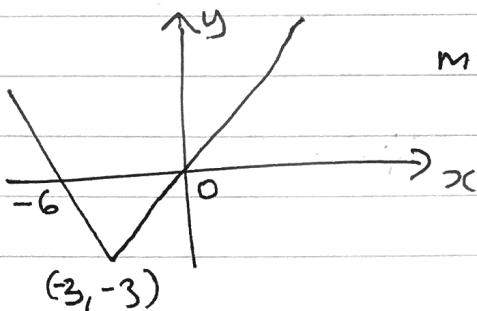
$$f(0.4735) = 0.001594 \dots$$

change in sign between $x = 0.4725$
and $x = 0.4735$ \therefore a root of $f(x)$
is 0.473 to 3 dp

Q4a)

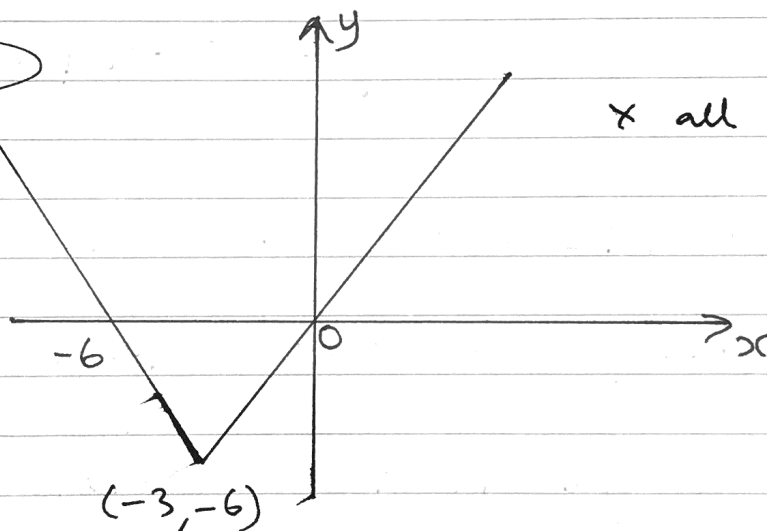


b) $f(x+5)$:



move everything 5 units to the left

$2f(x+5)$



x all y-values by 2

$$\bullet \text{ (Q5a)} \quad \frac{9(4+x)}{(4-3x)(4+3x)} = \frac{A}{4-3x} + \frac{B}{4+3x}$$

$$\therefore 9(4+x) = A(4+3x) + B(4-3x)$$

$$x = \frac{4}{3} : \quad 48 = 8A \quad \therefore A = 6 //$$

$$\underline{x = 0} : \quad 36 = 4(6) + 4B$$

$$\therefore B = 3 //$$

$$\text{so } \frac{9(4+x)}{(4-3x)(4+3x)} = \frac{6}{4-3x} + \frac{3}{4+3x}$$

$$\bullet \text{ b) } \int f(x) dx = 3 \int \left[\frac{2}{4-3x} + \frac{1}{4+3x} \right] dx$$

$$= 3 \left[-\frac{2}{3} \ln|4-3x| + \frac{1}{3} \ln|4+3x| \right] + c$$

$$= -2 \ln|4-3x| + \ln|4+3x| + c$$

$$= \ln \left| \frac{4+3x}{(4-3x)^2} \right| + \ln e^c$$

↖ a constant

$$= \ln \left(\frac{A(4+3x)}{(4-3x)^2} \right) \quad \text{where } A \text{ is an arbitrary constant.}$$

$$\text{Q6) Volume required} = \pi \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (3 \tan(\frac{x}{2})) dx$$

$$= 3\pi \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} [\tan(\frac{x}{2})] dx$$

$$= 3\pi \left[2 \ln(\sec \frac{x}{2}) \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$= 3\pi \left[2 \ln \left| \sec \frac{\pi}{4} \right| \right] - 3\pi \left[2 \ln \left| \sec \frac{\pi}{6} \right| \right]$$

$$= 6\pi \left[\ln \sqrt{2} - \ln \frac{2\sqrt{3}}{3} \right]$$

$$= 6\pi \times \ln \frac{\sqrt{6}}{2} = 3\pi \times \ln \left(\frac{\sqrt{6}}{2} \right)^2$$

$$= \boxed{3\pi \ln \frac{3}{2}}$$

$$(A = 3\pi)$$

● (Q7a) assuming l_1 and l_2 meet

$$\begin{pmatrix} 13 + 3\lambda \\ 15 + 3\lambda \\ -8 - 4\lambda \end{pmatrix} = \begin{pmatrix} 7 + 2\mu \\ -6 - 3\mu \\ 14 + 2\mu \end{pmatrix} \begin{array}{l} \sim \textcircled{1} \\ \sim \textcircled{2} \\ \sim \textcircled{3} \end{array}$$

$$\textcircled{3} : -8 - 4\lambda = 14 + 2\mu$$

$$\begin{aligned} \div 2 : \mu + 7 &= -4 - 2\lambda \\ \mu &= -11 - 2\lambda // \end{aligned}$$

$$\begin{aligned} \hookrightarrow \textcircled{2} : 15 + 3\lambda &= -6 - 3(-11 - 2\lambda) \\ 3\lambda &= 12 + 6\lambda \\ \lambda &= -4 // \end{aligned}$$

$$\begin{aligned} \text{sub } \lambda = -4 \text{ into } \textcircled{1} : 13 + 3(-4) &= 7 + 2\mu \\ \mu &= \frac{-6}{2} = -3 // \end{aligned}$$

$$\text{sub } \lambda = -4 \text{ into } \textcircled{3} : \mu = -11 - 2(-4) = -3 //$$

Values of μ and λ are consistent
so l_1 and l_2 do indeed intersect

$$B = \begin{pmatrix} 7 + 2(-3) \\ -6 - 3(-3) \\ 14 + 2(-3) \end{pmatrix} = \boxed{\begin{pmatrix} 1 \\ 3 \\ 8 \end{pmatrix}} //$$

b) using direction vectors:

$$\begin{pmatrix} 3 \\ 3 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} = 6 - 9 - 8 = -11 //$$

$$\left| \begin{pmatrix} 3 \\ 3 \\ -4 \end{pmatrix} \right| = \sqrt{3^2 + 3^2 + 4^2} = \sqrt{34}$$

$$\left| \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} \right| = \sqrt{2^2 + 3^2 + 2^2} = \sqrt{17}$$

$$\therefore \cos \theta = \frac{-11}{\sqrt{34} \times \sqrt{17}} = \frac{-11}{17\sqrt{2}}$$

$$\cos^{-1} \left(\frac{-11}{17\sqrt{2}} \right) = 117.23^\circ \dots$$

but we want the acute angle

$$180 - 117.23 \dots = \boxed{62.8^\circ}$$

c) assume A lies on l ...

$$\begin{pmatrix} -5 \\ -3 \\ 16 \end{pmatrix} = \begin{pmatrix} 13 + 3\lambda \\ 15 + 3\lambda \\ -8 - 4\lambda \end{pmatrix} \begin{matrix} \text{---} \textcircled{1} \\ \text{---} \textcircled{2} \\ \text{---} \textcircled{3} \end{matrix}$$

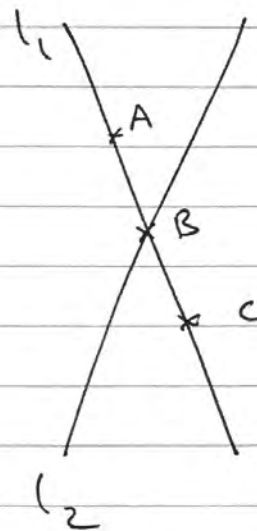
$$\begin{array}{l}
 \textcircled{1} : 3\lambda = -18 \quad \therefore \lambda = -6 \\
 \textcircled{2} : 3\lambda = -18 \quad \therefore \lambda = -6 \\
 \textcircled{3} : -4\lambda = 24 \quad \therefore \lambda = -6
 \end{array}
 \left. \vphantom{\begin{array}{l} \textcircled{1} \\ \textcircled{2} \\ \textcircled{3} \end{array}} \right\} \begin{array}{l} \text{Values of} \\ \lambda \text{ are} \\ \text{consistent} \\ \text{so A does} \\ \text{lie on } l_1. \end{array}$$

$$d) \vec{OC} = \vec{OA} + 2\vec{AB}$$

$$\vec{AB} = \vec{OB} - \vec{OA}$$

$$\vec{AB} = \begin{pmatrix} 1 \\ 3 \\ 8 \end{pmatrix} - \begin{pmatrix} -5 \\ -3 \\ 16 \end{pmatrix}$$

$$\vec{AB} = \begin{pmatrix} 6 \\ 6 \\ -8 \end{pmatrix}$$



$$\therefore \vec{OC} = \begin{pmatrix} -5 \\ -3 \\ 16 \end{pmatrix} + 2 \begin{pmatrix} 6 \\ 6 \\ -8 \end{pmatrix}$$

$$= \begin{pmatrix} -5 + 12 \\ -3 + 12 \\ 16 - 16 \end{pmatrix} = \begin{pmatrix} 7 \\ 9 \\ 0 \end{pmatrix}$$

$$Q8) \quad y = 8 \tan(2x)$$

$$\frac{dy}{dx} = 16 \sec^2 2x$$

$$\therefore \frac{dx}{dy} = \frac{1}{16 \sec^2 2x} //$$

$$\frac{y}{8} = \tan 2x$$

$$\left(\frac{y}{8}\right)^2 = \tan^2 2x$$

$$1 + \left(\frac{y}{8}\right)^2 = 1 + \tan^2 2x$$

$$1 + \tan^2 2x = \sec^2 2x$$

$$\therefore 1 + \left(\frac{y}{8}\right)^2 = 1 + \frac{y^2}{64} = \frac{64 + y^2}{64} = \sec^2 2x //$$

$$\therefore \frac{dx}{dy} = \frac{1}{16 \left(\frac{64 + y^2}{64}\right)} = \frac{1}{\frac{64 + y^2}{4}}$$

$$= \boxed{\frac{4}{64 + y^2}}$$

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$$\underline{1 + \cot^2 x = \operatorname{cosec}^2 x}$$

● Q9a) LHS = $\frac{\cot^2 x}{1 + \cot^2 x} = \frac{\operatorname{cosec}^2 x - 1}{\operatorname{cosec}^2 x}$

$$= \frac{\operatorname{cosec}^2 x}{\operatorname{cosec}^2 x} - \frac{1}{\operatorname{cosec}^2 x} \quad \leftarrow \sin^2 x$$

$$= 1 - \sin^2 x$$

$$= \cos^2 x \quad \text{as required.}$$

b) $\cos^2 x = 8 \cos 2x + 2 \cos x$

$$\cos^2 x = 8(2 \cos^2 x - 1) + 2 \cos x$$

$$16 \cos^2 x - 8 + 2 \cos x - \cos^2 x = 0$$

$$15 \cos^2 x - \cos^2 x + 2 \cos x - 8 = 0$$

$$15 \cos^2 x + 2 \cos x - 8 = 0$$

By Quadratic Formula:

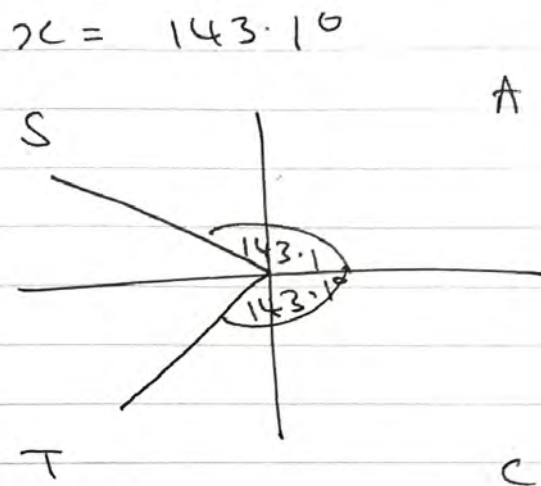
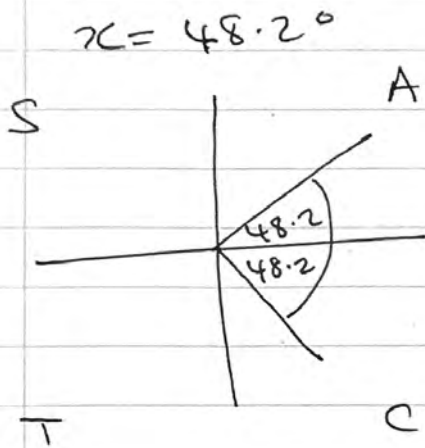
$$\cos x = \frac{2}{3}, \quad |$$

$$\cos x = -\frac{4}{5}$$

$$x = \cos^{-1}\left(\frac{2}{3}\right) = 48.2^\circ$$

$$x = \cos^{-1}\left(-\frac{4}{5}\right) = 143.1^\circ$$

$$0 \leq \alpha < 360^\circ$$

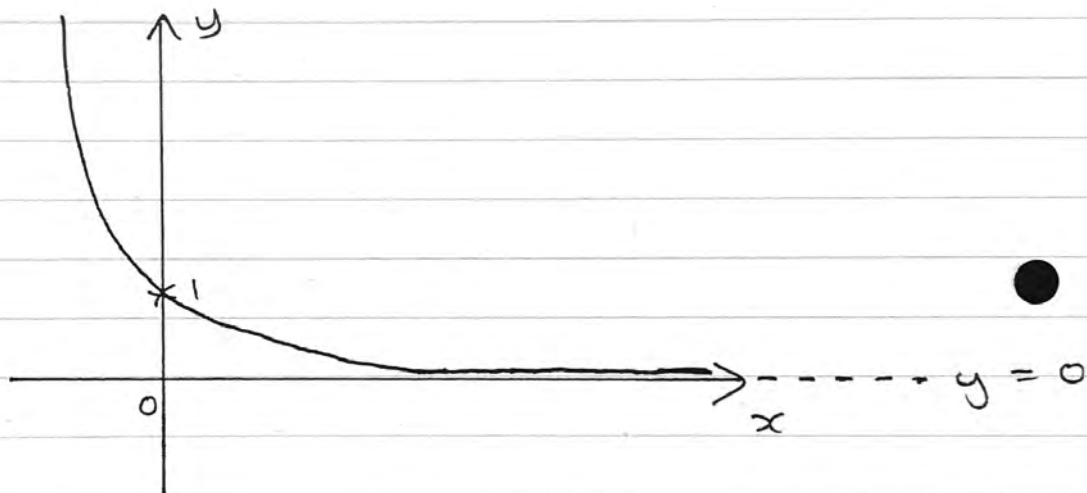


$$\alpha = 48.2^\circ, 360 - 48.2^\circ$$

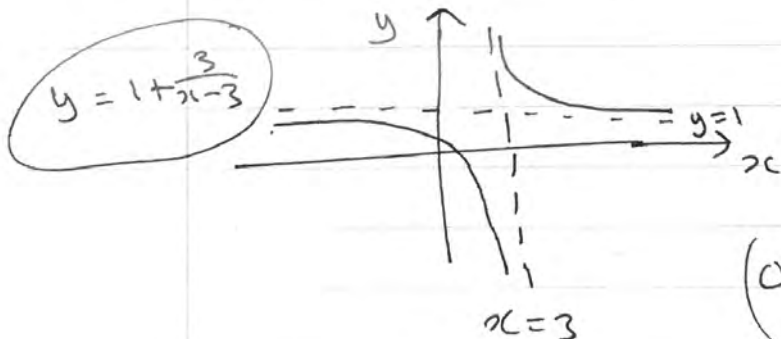
$$\alpha = 143.1^\circ, 360 - 143.1^\circ$$

$$\alpha = 48.2^\circ, 143.1^\circ, 216.9^\circ, 311.8^\circ$$

Q10 a)



$$b) \quad \frac{\alpha}{\alpha-3} = \frac{\alpha-3+3}{\alpha-3} = 1 + \frac{3}{\alpha-3}$$



so range is $y > 1$

(or by observation of the equation)

$$c) \quad y = \frac{x}{x-3}$$

$$x \leftrightarrow y; \quad x = \frac{y}{y-3}$$

now make y the subject again:

$$xy - 3x = y$$

$$y(x-1) = 3x$$

$$\therefore y = \frac{3x}{x-1} = g^{-1}(x) //$$

domain of $g^{-1}(x) = \text{range of } g(x)$

\therefore $x > 1$ is the domain.

$$d) \quad fg(x) = f[g(x)] = e^{-2\left(\frac{x}{x-3}\right)}$$

$$= e^{\frac{-2x}{x-3}}$$

$$\underline{fg(x) = 3} : e^{\frac{-2x}{x-3}} = 3$$

$$\Rightarrow \ln(3) = \ln\left(e^{\frac{-2x}{x-3}}\right)$$

$$\Rightarrow \frac{-2x}{x-3} = \ln 3$$

$$\Rightarrow x \ln 3 - 3 \ln 3 = -2x$$

$$\Rightarrow x(\ln 3 + 2) = 3 \ln 3$$

$$\Rightarrow \boxed{x = \frac{3 \ln 3}{2 + \ln 3}}$$

at A x is minimum
 at B x is max.

Q11a) at A/B, x is a max/min, $x = 3 \cos t$

$$\text{hence } \left. \begin{array}{l} x_{\max} = 3 \\ x_{\min} = -3 \end{array} \right\} \text{ as } -1 \leq \cos t \leq 1$$

$$\text{so } \boxed{A(-3, 0)} \text{ and } \boxed{B(3, 0)}$$

b) $x = 0$ at $t = \frac{\pi}{2}, \frac{3\pi}{2}$

$$y = 0 \text{ at } t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi.$$

only values of t where both $x=0$ and $y=0$
 are $\boxed{\frac{\pi}{2}}$ and $\boxed{\frac{3\pi}{2}}$

$$\begin{array}{l} \text{c) } x = 3 \cos t \rightarrow \frac{dx}{dt} = -3 \sin t \\ y = 9 \sin 2t \rightarrow \frac{dy}{dt} = 18 \cos 2t \end{array} \left. \vphantom{\begin{array}{l} x = 3 \cos t \\ y = 9 \sin 2t \end{array}} \right\} \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\therefore \frac{dy}{dx} = \frac{18 \cos 2t}{-3 \sin t}$$

$$= \frac{18 \cos \frac{\pi}{3}}{-3 \sin \frac{\pi}{6}} = \boxed{-6}$$

$$d) \quad y = 18 \sin t \cos t$$

$$x = 3 \cos t$$

$$\therefore y = \frac{18 \sin t}{3} \times x \quad //$$

$$x = 3 \cos t$$

$$\frac{x}{3} = \cos t$$

$$\left(\frac{x}{3}\right)^2 = \cos^2 t$$

$$1 - \left(\frac{x}{3}\right)^2 = 1 - \cos^2 t = \sin^2 t$$

$$\therefore 1 - \frac{x^2}{9} = \sin^2 t = \frac{9 - x^2}{9}$$

$$\therefore \sin t = \frac{\sqrt{9 - x^2}}{3} \quad //$$

$$\text{so } y = 6x \times \frac{\sqrt{9 - x^2}}{3} = 2x \sqrt{9 - x^2}$$

$$y^2 = (2x \sqrt{9 - x^2})^2$$

$$\Rightarrow \boxed{y^2 = 4x^2(9 - x^2)}$$

$$\bullet \text{ Q(2a)} \quad 2\sin x - 4\cos x \equiv R\sin(x-d) \equiv R\sin x \cos d - R\cos x \sin d$$

Comparing coefficients: $2 = R\cos d \sim \textcircled{1}$

$$4 = R\sin d \sim \textcircled{2}$$

$$\frac{\textcircled{2}}{\textcircled{1}}: \frac{R\sin d}{R\cos d} = \tan d = \frac{4}{2} = 2 //$$

$$\therefore d = \tan^{-1}(2) = \boxed{1.11^\circ}$$

Finding R: $R = \sqrt{2^2 + 4^2} = \boxed{2\sqrt{5}}$

$$\therefore 2\sin x - 4\cos x \equiv \boxed{2\sqrt{5} \sin(x - 1.11)}$$

$$\bullet \text{ b) } H = 12 + 2 \left(2\sqrt{5} \sin \left(\frac{2\pi t}{365} - 1.11 \right) \right)$$

$$H_{\max} \text{ occurs when } \sin \left(\frac{2\pi t}{365} - 1.11 \right) = 1$$

$$\therefore H_{\max} = 12 + 4\sqrt{5}(1) = \boxed{20.9} \text{ hours}$$

$$H_{\min} \text{ occurs when } \sin \left(\frac{2\pi t}{365} - 1.11 \right) = -1$$

$$\therefore H_{\min} = 12 - 4\sqrt{5} = \boxed{3.06} \text{ hours}$$

$$c) \quad 17 = 12 + 4\sqrt{5} \sin\left(\frac{2\pi t}{365} - 1.11\right)$$

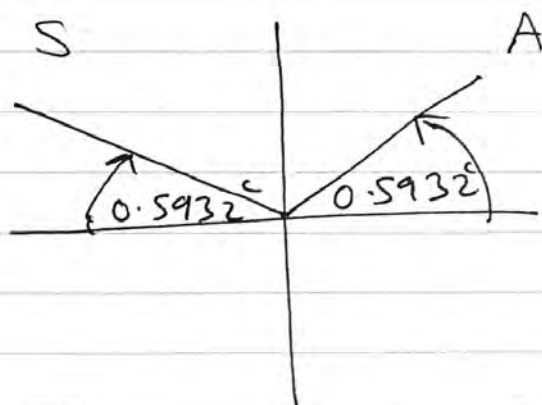
$$\frac{5}{4\sqrt{5}} = \frac{\sqrt{5}}{4} = \sin\left(\frac{2\pi t}{365} - 1.11\right)$$

$$\frac{2\pi t}{365} - 1.11 = \sin^{-1}\left(\frac{\sqrt{5}}{4}\right) = 0.5932$$

$$0 \leq \frac{2\pi t}{365} \leq 2\pi$$

||
↓

$$\left(-1.11 \leq \frac{2\pi t}{365} - 1.11 \leq 2\pi - 1.11\right) \quad T \quad C$$



$$\therefore \frac{2\pi t}{365} - 1.11 = 0.593, \quad 2.548$$

$$\frac{2\pi t}{365} = 1.703, \quad 3.658$$

$$t = 98.9\dots, \quad 212.5\dots$$

So $t = 99$ days and 213 days

● (Q13a) $h = \frac{b-a}{n} = \frac{5e - e}{4} = e$

$\therefore \text{Area} \approx \frac{1}{2} \times e [0.3114 + 0.1215 + 2(0.2195 + 0.1712 + 0.1416)]$
 $\approx \boxed{2.04}$ to 3 s.f.

● b) $\int \frac{1}{2x} \ln 2x \, dx$ $\left[\begin{array}{l} u = \ln 2x \\ \frac{du}{dx} = \frac{2}{2x} = \frac{1}{x} \\ \therefore dx = x \, du \end{array} \right]$

$\Rightarrow \int \left[\frac{1}{2x} \times u \times x \right] du = \int \left[\frac{1}{2} u \right] du$

$= \frac{u^2}{4} + c = \boxed{\frac{(\ln 2x)^2}{4} + c}$

● c) $R = \left[\frac{(\ln 2x)^2}{4} \right]_e^{5e} = \left[\frac{(\ln 10e)^2}{4} \right] - \left[\frac{(\ln 2e)^2}{4} \right]$

$= \frac{(\ln(10e))^2 - (\ln(2e))^2}{4} = \boxed{2.01}$

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By QUOTIENT RULE

$$d) \quad y = \frac{1}{2x} \ln 2x = \frac{\ln 2x}{2x}$$

$$u = \ln 2x \quad \longrightarrow \quad u' = \frac{1}{x}$$

$$v = 2x \quad \longrightarrow \quad v' = 2$$

$$\frac{dy}{dx} = \frac{\frac{2x}{x} - 2 \ln 2x}{(2x)^2} \quad \left(= \frac{vu' - uv'}{v^2} \right)$$

$$= \frac{2 - \ln(4x^2)}{4x^2} = \frac{2 - \ln(e^4)}{e^4}$$

\curvearrowright
 at $x = \frac{e^2}{2}$

$$= -\frac{2}{e^4} //$$

$$\text{at } x = \frac{e^2}{2}, \quad y = \frac{1}{e^2} \ln e^2 = \frac{2}{e^2} //$$

$$\therefore y - \frac{2}{e^2} = -\frac{2}{e^4} \left(x - \frac{e^2}{2} \right)$$

$$\Rightarrow y = -\frac{2x}{e^4} + \frac{1}{e^2} + \frac{2}{e^2}$$

$$\Rightarrow \boxed{y = -\frac{2x}{e^4} + \frac{3}{e^2}}$$

$$Q14a) \quad V = \frac{4}{3} \pi r^3$$

$$\frac{dV}{dr} = \boxed{4\pi r^2}$$

$$b) \quad \frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$$

$$\frac{dr}{dV} = \frac{1}{4\pi r^2} \quad \text{and} \quad \frac{dV}{dt} = \frac{9000\pi}{(t+81)^{5/4}}$$

$$\therefore \frac{dr}{dt} = \frac{9000\pi}{(t+81)^{5/4} \times 4\pi r^2} = \boxed{\frac{2250}{r^2(t+81)^{5/4}}}$$

$$c) \quad \frac{dr}{dt} = \frac{2250}{r^2(t+81)^{5/4}}$$

$$r^2 \frac{dr}{dt} = \frac{2250}{(t+81)^{5/4}}$$

$$\int (r^2) dr = 2250 \int (t+81)^{-5/4} dt$$

$$\Rightarrow \frac{r^3}{3} = 2250 \left[\frac{(t+81)^{-1/4}}{-1/4} \right] + C$$

$$\Rightarrow \frac{r^3}{3} = \frac{-9000}{(t+81)^{1/4}} + C //$$

$$\underline{t=0, r=3} : 9 = \frac{-9000}{81^{1/4}} + c$$

$$c = 3009 \quad \underline{\underline{=}}$$

$$\therefore \frac{r^3}{3} = \frac{-9000}{(t+81)^{1/4}} + 3009$$

$$r^3 = \frac{-27000}{\sqrt[4]{t+81}} + 9027$$

$$r = \left[9027 - \frac{27000}{\sqrt[4]{t+81}} \right]^{1/3}$$

$$d) \underline{t=175} : r = \left[9027 - \frac{27000}{\sqrt[4]{256}} \right]^{1/3} = \boxed{13.2}$$

$$e) \frac{dr}{dt} = \frac{2250}{r^2(t+81)^{5/4}} \quad \text{from (c)}$$

$$\text{at } t=175, \quad \frac{dr}{dt} = \frac{2250}{(13.2)^2(175+81)^{5/4}} = \boxed{0.0126} \text{ cm/s.}$$