

(34 Jan 2017 (MA))

$$Q1) \frac{d}{dx} [x^3 + 3x^2y + y^3 = 37]$$

$$\Rightarrow 3x^2 + 3x^2 \frac{dy}{dx} + 6xy + 3y^2 \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} (3x^2 + 3y^2) = -3x^2 - 6xy$$

$$\Rightarrow \frac{dy}{dx} = \frac{-3(x^2 + 2xy)}{3x^2 + 3y^2} = \frac{-(x^2 + 2xy)}{x^2 + y^2} //$$

$$\text{at } (1, 3), \frac{dy}{dx} = \frac{-(1 + 2(3))}{1 + 9} = \frac{-7}{10} //$$

$$\Rightarrow y - 3 = \frac{-7}{10} (x - 1)$$

$$\Rightarrow y - 3 = \frac{-7}{10} x + \frac{7}{10}$$

$$\Rightarrow y = \frac{-7}{10} x + \frac{37}{10}$$

$$\stackrel{\times 10}{\Rightarrow} 10y = -7x + 37$$

$$\Rightarrow 7x + 10y - 37 = 0$$

$$Q2a) f(x) = x^3 - 5x + 16 = 0$$

$$x^3 = 5x - 16$$

$$x = (5x - 16)^{\frac{1}{3}} \quad // \quad \begin{array}{l} a=5 \\ b=-16 \end{array}$$

$$b) \quad x_1 = -3$$

$$x_2 = (5(-3) - 16)^{\frac{1}{3}} \approx -3.1414 \dots$$

$$x_3 = (5(-3.1414) - 16)^{\frac{1}{3}} \approx -3.16508 \dots$$

$$x_4 = (5(-3.16508) - 16)^{\frac{1}{3}} \approx -3.16902 \dots$$

$$\text{So } \boxed{\begin{array}{l} x_1 = -3.141 \\ x_2 = -3.165 \\ x_3 = -3.169 \end{array}}$$

$$c) \quad f(-3.165) = (-3.165)^3 - 5(-3.165) + 16 = 0.12048 \dots$$

$$f(-3.175) = (-3.175)^3 - 5(-3.175) + 16 = -0.13098 \dots$$

change in sign between $x = -3.165$
and $x = -3.175$ \therefore a root α lies
between these values hence $\alpha = -3.17$
to 2 d.p.

$$\text{Q3a)} \quad \frac{9+11x}{(1-x)(3+2x)} = \frac{A}{1-x} + \frac{B}{3+2x}$$

$$9+11x = A(3+2x) + B(1-x)$$

$$\underline{x=1} : 20 = 5A \quad \therefore A = 4 //$$

$$\underline{x=0} : 9 = 3A + B \quad \therefore B = 9 - 12 = -3 //$$

$$\text{so } \frac{9+11x}{(1-x)(3+2x)} = \frac{4}{1-x} - \frac{3}{3+2x} //$$

①
②

$$\text{b) } \textcircled{1} : 4(1-x)^{-1} \underset{\substack{[n=-1] \\ x=-x}}{\approx} \left[1 + x + \frac{-1(-2)}{2}(-x)^2 + \frac{-1(-2)(-3)}{6}(-x)^3 \right] \times 4$$

$$\approx 4 \times [1 + x + x^2 + x^3]$$

$$\approx 4 + 4x + 4x^2 + 4x^3 //$$

$$\textcircled{2} : 3(3+2x)^{-1} = 3 \cdot 3^{-1} (1 + \frac{2}{3}x)^{-1} = (1 + \frac{2}{3}x)^{-1} //$$

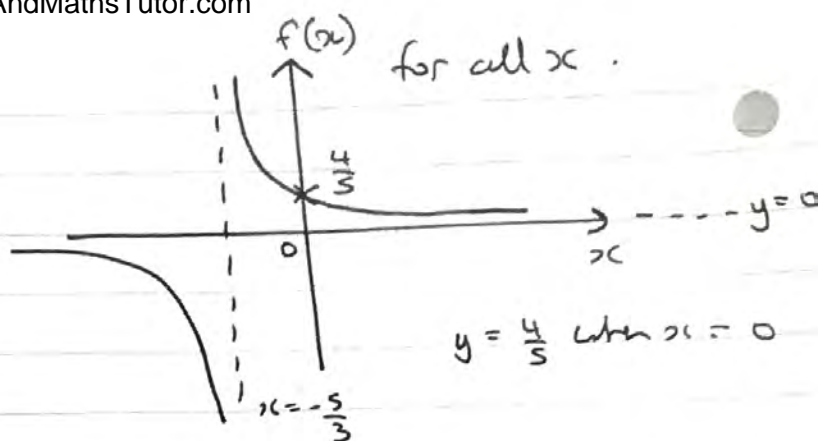
$$\Rightarrow (1 + \frac{2}{3}x)^{-1} \underset{\substack{[x = \frac{2}{3}x] \\ [n = -1]}}{\approx} 1 - \frac{2}{3}x + \frac{-1(-2)}{2}(\frac{2}{3}x)^2 + \frac{-1(-2)(-3)}{6}(\frac{2}{3}x)^3$$

$$\approx 1 - \frac{2}{3}x + \frac{4}{9}x^2 - \frac{8}{27}x^3 //$$

$$\therefore \frac{9+11x}{(1-x)(3+2x)} \approx \frac{4 + 4x + 4x^2 + 4x^3}{1 - \frac{2}{3}x + \frac{4}{9}x^2 - \frac{8}{27}x^3}$$

$$\approx \boxed{3 + \frac{14}{3}x + \frac{32}{9}x^2 + \frac{116}{27}x^3}$$

Q1a) $0 < x < \frac{4}{5}$



b) $y = \frac{4}{3x+5}$

$x \leftrightarrow y$; $x = \frac{4}{3y+5}$

make y subject: $(3y+5)x = 4$

$$3xy + 5x = 4$$

$$y = \frac{4 - 5x}{3x} //$$

c) $g(x) = \frac{1}{x}$

$$\therefore f[g(x)] = \frac{4}{3\left(\frac{1}{x}\right) + 5} = \frac{4}{\frac{3}{x} + 5} = \boxed{\frac{4x}{3+5x}}$$

d) $fg(x) = \frac{4x}{3x+5}$

$$gf(x) = \frac{1}{\frac{4}{3x+5}} = \frac{3x+5}{4} //$$

$$\Rightarrow \frac{3x+5}{4} = \frac{4x}{3x+5}$$

$$\Rightarrow (3x+5)^2 = 16x$$

$$\Rightarrow 9x^2 + 30x + 25 + 16x = 0$$

$$\Rightarrow 9x^2 + 46x + 25 = 0$$

$$b^2 - 4ac = 46^2 - 4(9)(25) = -704 < 0$$

\therefore no real solutions exist.

Q5a)

x	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π	$\frac{9\pi}{4}$	$\frac{5\pi}{2}$
y	0	$\frac{7\pi\sqrt{2}}{8}$	2π	$\frac{9\pi\sqrt{2}}{8}$	0

$$\cos \frac{7\pi}{4} = \frac{\sqrt{2}}{2} \rightarrow x \cos x = \frac{7\pi}{4} \cdot \frac{\sqrt{2}}{2} = \frac{7\pi\sqrt{2}}{8} //$$

$$\cos \frac{9\pi}{4} = \frac{\sqrt{2}}{2} \rightarrow x \cos x = \frac{9\pi}{4} \cdot \frac{\sqrt{2}}{2} = \frac{9\pi\sqrt{2}}{8} //$$

b) $n = \frac{b-a}{h} = \frac{\frac{5\pi}{2} - \frac{3\pi}{2}}{\frac{\pi}{4}} = \frac{\pi}{\frac{\pi}{4}} //$ (5 values so $n=4$)

$$\text{Area} \approx \frac{1}{2} \times \frac{\pi}{4} \left[0 + 0 + 2 \left(\frac{7\pi\sqrt{2}}{8} + 2\pi + \frac{9\pi\sqrt{2}}{8} \right) \right]$$

$$\approx \boxed{11.91} \text{ units}^2.$$

c) $\int (x \cos x) dx$ — By parts

$u = x$	$v' = \cos x$
$u' = 1$	$v = \sin x$

$$\Rightarrow [x \sin x] - \int (\sin x) dx$$

$$\Rightarrow \underline{\underline{x \sin x + \cos x + C}} = \int (x \cos x) dx.$$

d) $\left[x \sin x + \cos x \right]_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} = \left[\frac{5\pi}{2} (1) + (0) \right] - \left[-\frac{3\pi}{2} + 0 \right]$

$$= \frac{5\pi}{2} + \frac{3\pi}{2} = \boxed{4\pi}$$

(Q6i)

$$y = 5x^2 \ln 3x$$

PRODUCT RULE

$$\frac{dy}{dx} = 10x \ln 3x + 5x^2 \times \frac{1}{x}$$

$$= 10x \ln 3x + 5x = \boxed{5x(1 + 2 \ln 3x)}$$

ii)

$$y = \frac{x}{\sin x + \cos x}$$

QUOTIENT RULE

$$\Downarrow$$

$$u = x \quad u' = 1$$

$$v = (\sin x + \cos x) \quad v' = \cos x - \sin x$$

$$\frac{dy}{dx} = \frac{\sin x + \cos x - x(\cos x - \sin x)}{(\sin x + \cos x)^2}$$

$$= \frac{\sin x + x \sin x + \cos x - x \cos x}{(\sin x + \cos x)^2}$$

$$= \frac{(\sin x + \cos x) + x(\sin x - \cos x)}{(\sin x + \cos x)^2}$$

$$= \frac{\sin x(1 + x) + \cos x(1 - x)}{(\sin x + \cos x)^2}$$

□

Q7ai) $y = |ax + b|$

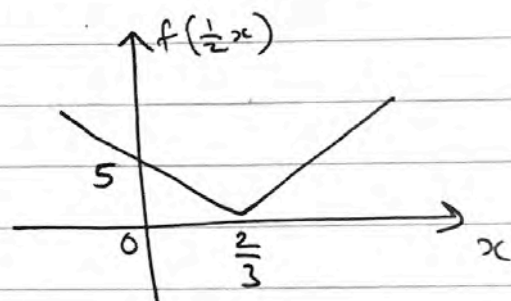
$$\left(\frac{1}{3}, 0\right): 0 = \left|\frac{1}{3}a + b\right|$$

$$\Rightarrow a + 3b = 0 //$$

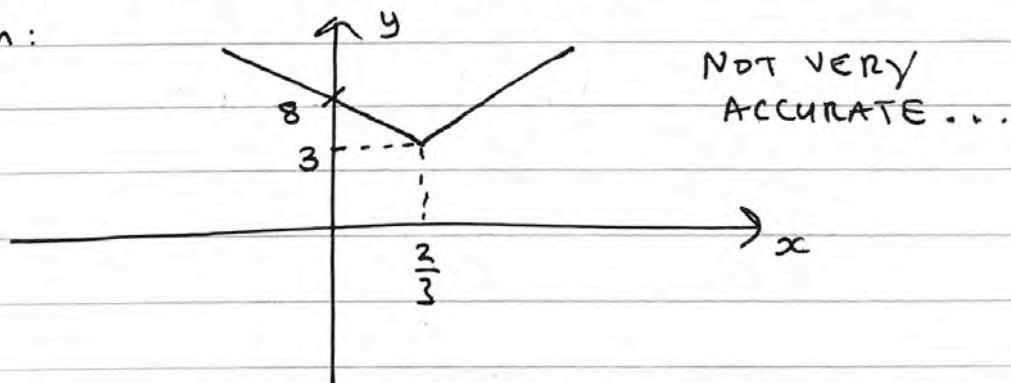
$$(0, 5): 5 = |b| \left[\begin{array}{l} \therefore b = 5 \\ \downarrow \\ a = -3(5) = -15 \end{array} \right] \text{ or } \left[\begin{array}{l} b = -5 \\ \downarrow \\ a = -3(-5) = 15 \end{array} \right] //$$

ii) $y = |-15x + 5|$ or $y = |15x - 5|$

b) $f\left(\frac{1}{2}x\right)$ means multiply all x -values by 2.



$f\left(\frac{1}{2}x\right) + 3$ means multiply all x by 2
AND translation of 3 in the positive
 y -direction:



$$\text{Q8a)} \quad \tan(2x+x) = \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$$

$$= \frac{\left(\frac{2 \tan x}{1 - \tan^2 x}\right) + \tan x}{1 - \left(\frac{2 \tan^3 x}{1 - \tan^2 x}\right)} \times (1 - \tan^2 x)$$

$$= \frac{2 \tan x + \tan x (1 - \tan^2 x)}{(1 - \tan^2 x) - 2 \tan^2 x} = \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}$$

$$= \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}$$

$$\text{b)} \quad \text{LHS} = \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x} = 1 \tan x$$

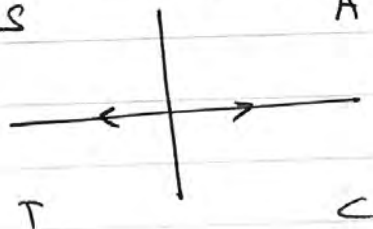
$$\Rightarrow 1 \tan x (1 - 3 \tan^2 x) = 3 \tan x - \tan^3 x$$

$$\Rightarrow 1 \tan x - 3 \tan^3 x + \tan^3 x - 3 \tan x = 0$$

$$\Rightarrow \tan x (8 - 32 \tan^2 x) = 0 //$$

$$\tan x = 0$$

$$x = \tan^{-1}(0) = \boxed{0^\circ} //$$



(no other solutions)

$$8 - 32 \tan^2 x = 0$$

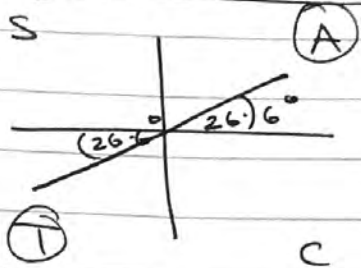
$$\tan^2 x = \frac{8}{32}$$

$$\tan x = \pm \sqrt{\frac{8}{32}}$$

$$-30 < x < 30^\circ$$

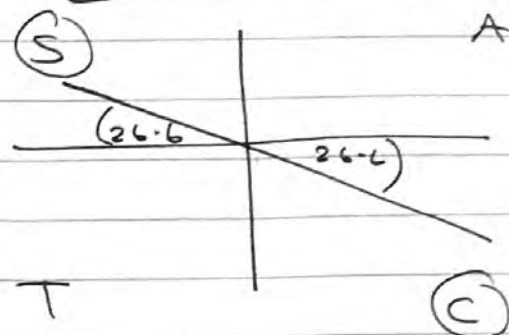
$$\tan x = \sqrt{\frac{8}{32}}$$

$$x = 26.6^\circ$$



$$\tan x = -\sqrt{\frac{8}{32}}$$

$$x = -26.6^\circ$$



Q9a)

$$\int_0^{12} \left[\frac{x}{(2x+3)^2} \right] dx$$

$$u = 2x+3$$

$$\frac{du}{dx} = 2 \quad \therefore dx = \frac{1}{2} du //$$

$$u-3=2x$$

$$\therefore x = \frac{1}{2}(u-3) //$$

x	u
0	3
12	27

$$2(0)+3=3 //$$

$$2(12)+3=27 //$$

$$\Rightarrow \int_3^{27} \left[\frac{\frac{1}{2}(u-3)}{u^2} \times \frac{1}{2} \right] du$$

$$\Rightarrow \frac{1}{4} \int_3^{27} \left[\frac{u-3}{u^2} \right] du = \frac{1}{4} \int_3^{27} \left[\frac{1}{u} - \frac{3}{u^2} \right] du$$

$$\Rightarrow \frac{1}{4} \int_3^{27} [u^{-1} - 3u^{-2}] du = \frac{1}{4} \left[\ln|u| - \frac{3u^{-1}}{-1} \right]_3^{27}$$

$$\Rightarrow \frac{1}{4} \left[\ln|u| + \frac{3}{u} \right]_3^{27} = \frac{1}{4} \left[\ln 27 + \frac{3}{27} \right] - \frac{1}{4} \left[\ln 3 + 1 \right]$$

$$\Rightarrow \frac{1}{4} [\ln 27 - \ln 3] + \frac{1}{4} \left[\frac{1}{9} - 1 \right] = \frac{1}{4} \ln 9 - \frac{2}{9}$$

$$\frac{1}{4} \ln 9 - \frac{2}{9} = \frac{1}{2} \ln(\sqrt{9}) - \frac{2}{9}$$

$$= \boxed{\frac{1}{2} \ln 3 - \frac{2}{9}}$$

$$b) V = \pi \int_0^{12} y^2 dx = 81\pi \int_0^{12} \frac{1}{(2x+3)^2} dx = 81\pi \times \text{answer from (a)}$$

$$\therefore V = 81\pi \left[\frac{1}{2} \ln 3 - \frac{2}{9} \right]$$

$$= \frac{81\pi}{2} \ln 3 - 18\pi \quad (\text{o.e.})$$

$$10a) \underline{t=0}: N = \frac{300}{3+17} = \frac{300}{20} = \boxed{15}$$

$$b) \underline{t=10}: N = \frac{300}{3+17e^{-2}} = 56.5963 \dots$$

∴ number of insects = $\boxed{56}$
(not quite at 57!)
so round down.

$$c) \underline{82=N}: 82 = \frac{300}{3+17e^{-0.2t}}$$

$$\Rightarrow \frac{300}{82} = 3+17e^{-0.2t}$$

$$\Rightarrow \frac{\frac{300}{82} - 3}{17} = e^{-0.2t} = \frac{27}{697} //$$

$$\Rightarrow -0.2t = \ln\left(\frac{27}{697}\right)$$

$$\Rightarrow t = \frac{\ln\left(\frac{27}{697}\right)}{-0.2} = \boxed{16.3 \text{ weeks}}$$

$$d) N = 300(3 + 17e^{-0.2t})^{-1}$$

$$\therefore \text{PRODUCT RULE: } \frac{dN}{dt} = -300(3 + 17e^{-0.2t})^{-2} \times (-0.2 \times 17e^{-0.2t})$$

$$= \frac{60e^{-0.2t}}{(3 + 17e^{-0.2t})^2} \times 17$$

$$= \frac{1020e^{-0.2t}}{(3 + 17e^{-0.2t})^2}$$

//

$$\underline{t=5}: \frac{dN}{dt} = \frac{1020e^{-1}}{(3 + 17e^{-1})^2} = 4.382 \dots$$

$$= \boxed{4} \text{ insects per week}$$

$$Q11a) 35\sin x - 12\cos x \equiv R\sin(x-d) \equiv R\sin x \cos d - R\cos x \sin d$$

$$\underline{\text{Comparing coefficients:}} \quad 35 = R\cos d \quad \text{--- (1)}$$

$$12 = R\sin d \quad \text{--- (2)}$$

$$\frac{(2)}{(1)}: \frac{R\sin d}{R\cos d} = \tan d = \frac{12}{35}$$

$$\therefore d = \tan^{-1}\left(\frac{12}{35}\right) = \boxed{0.3303^\circ}$$

$$\underline{\text{Finding } R}: R = \sqrt{35^2 + 12^2} = \boxed{37}$$

$$\therefore 35\sin x - 12\cos x \equiv \boxed{37\sin(x - 0.3303^\circ)}$$

$$b) 70 \sin x - 24 \cos x = 2(35 \sin x - 12 \cos x)$$

$$\Rightarrow 74 \sin(x - 0.3303^\circ) = 37$$

$\div 74$

$$\Rightarrow \sin(x - 0.3303^\circ) = \frac{1}{2}$$

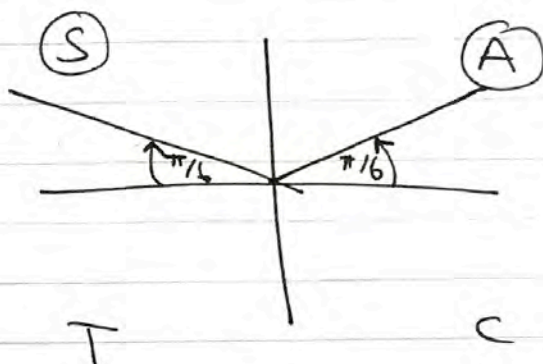
$$\Rightarrow x - 0.3303^\circ = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6} //$$

Solving in: $0 \leq x < 2\pi$

$$x = 0.3303^\circ = \frac{\pi}{6}, \left(\pi - \frac{\pi}{6}\right)$$

$$x - 0.3303^\circ = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\boxed{x = 0.854^\circ, 2.95^\circ}$$



$$c) y = \frac{7000}{31 + (37 \sin(x - 0.3303^\circ))^2}$$

Minimum value of y occurs when the denominator is the largest. (ie when $\sin(x - 0.3303^\circ) = 1$)

$$\Rightarrow y_{\min} = \frac{7000}{31 + (37)^2} = \boxed{5}$$

ii) first maximum occurs at $\left(\frac{\pi}{2} = x\right)$
for $y = \sin x$.

$$\Rightarrow x - 0.3303^\circ = \frac{\pi}{2}$$

$$\Rightarrow x = 0.3303 + \frac{\pi}{2} = \boxed{1.901^\circ}$$

$$\text{Q12a)} \quad \left. \begin{array}{l} t=0, x=0 \\ t=2, x=1.5 \end{array} \right\} \quad x = \frac{3}{4}t \quad \rightarrow \quad \boxed{t = \frac{4}{3}x}$$

$$\left(\frac{1.5}{2} = \frac{3}{4} \right)$$

$$\text{b)} \quad \underline{x=3} : \quad 3 = \frac{3}{4}t \quad \rightarrow \quad t = \frac{12}{3} = \boxed{4\text{s}}$$

$$\text{c)} \quad \frac{dx}{dt} = \frac{\lambda}{2x+1}$$

$$(2x+1) \frac{dx}{dt} = \lambda$$

$$\int (2x+1) dx = \int (\lambda) dt$$

$$x^2 + x = \lambda t + c$$

$$\underline{t=0, x=0} : \quad c = 0 //$$

$$\therefore \frac{x(x+1)}{\lambda} = t //$$

$$\text{d)} \quad \underline{x=1.5, t=2} : \quad \frac{1.5(2.5)}{\lambda} = 2$$

$$\lambda = \frac{1.5 \times 2.5}{2} = \boxed{\frac{15}{8}}$$

$$\text{e)} \quad t = \frac{3(3+1)}{\frac{15}{8}} = 6.4 \text{ hours after 4pm} //$$

$$= \boxed{10:24 \text{ pm}}$$

$$Q13a) \quad x=0: \quad 1 + \sqrt{3} \tan \theta = 0$$

$$\tan \theta = \frac{-1}{\sqrt{3}}$$

$$\theta = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = \frac{-\pi}{6} //$$

$$y = \frac{5}{\cos\left(\frac{-\pi}{6}\right)} = \frac{10\sqrt{3}}{3} //$$

$$\left. \vphantom{y} \right\} \boxed{\left(0, \frac{10\sqrt{3}}{3}\right)} \leftarrow A$$

$$b) \quad \frac{dx}{d\theta} = \sqrt{3} \sec^2 \theta \qquad \frac{dy}{d\theta} = 5 \sec \theta \tan \theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{5 \sec \theta \tan \theta}{\sqrt{3} \sec^2 \theta} = \frac{5 \tan \theta}{\sqrt{3} \cos \theta}$$

$$= \frac{5 \sin \theta}{\sqrt{3} \cos \theta} = \frac{5}{\sqrt{3}} \sin \theta //$$

$$\boxed{\therefore \lambda = \frac{5}{\sqrt{3}} = \frac{5\sqrt{3}}{3}}$$

$$c) \quad \frac{dy}{dx} = 0 \text{ at B: } \frac{5}{\sqrt{3}} \sin \theta = 0$$

$$\theta = 0 //$$

$$\therefore x_B = 1 + \sqrt{3} \tan(0) = 1 \quad \left. \vphantom{x_B} \right\} B(1, 5)$$

$$y_B = 5 \sec(0) = 5 //$$

$$d) \quad x = 1 + \sqrt{3} \tan \theta$$

$$\frac{(x-1)}{\sqrt{3}} = \tan \theta \quad \therefore \frac{(x-1)^2}{3} = \tan^2 \theta$$

$$\Rightarrow \frac{(x-1)^2}{3} = \sec^2 \theta - 1$$

$$\Rightarrow \frac{(x-1)^2}{3} + 1 = \sec^2 \theta //$$

cont.) $\sec \theta = \frac{\sqrt{(x-1)^2 + 3}}{\sqrt{3}} = \frac{1}{\sqrt{3}} \sqrt{x^2 - 2x + 1 + 3}$

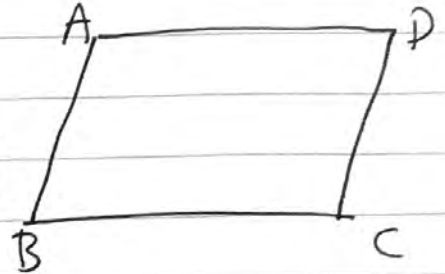
$$\therefore \sec \theta = \frac{5}{\sqrt{3}} \sqrt{x^2 - 2x + 4} = y$$

Q14a) $\vec{BC} = \vec{AD}$

$$\begin{pmatrix} -1 \\ 3 \\ 6 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 6 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$\vec{OC} - \vec{OB} = \vec{OD} - \vec{OA}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \\ -2 \end{pmatrix} = \vec{OB} = d$$



b) $\vec{AB} = \begin{pmatrix} 3 \\ -1 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 8 \end{pmatrix} \rightarrow |\vec{AB}| = \sqrt{2^2 + 2^2 + 8^2} = 6\sqrt{2}$

$\vec{CB} = \begin{pmatrix} 3 \\ -1 \\ 6 \end{pmatrix} - \begin{pmatrix} -1 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ 0 \end{pmatrix} \rightarrow |\vec{CB}| = \sqrt{4^2 + 4^2} = 4\sqrt{2}$

Both vectors must either be going towards B or from B. using \vec{BC} and \vec{BA} would also be ok.

$$\vec{AB} \cdot \vec{CB} = \begin{pmatrix} 2 \\ -2 \\ 8 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -4 \\ 0 \end{pmatrix} = 16$$

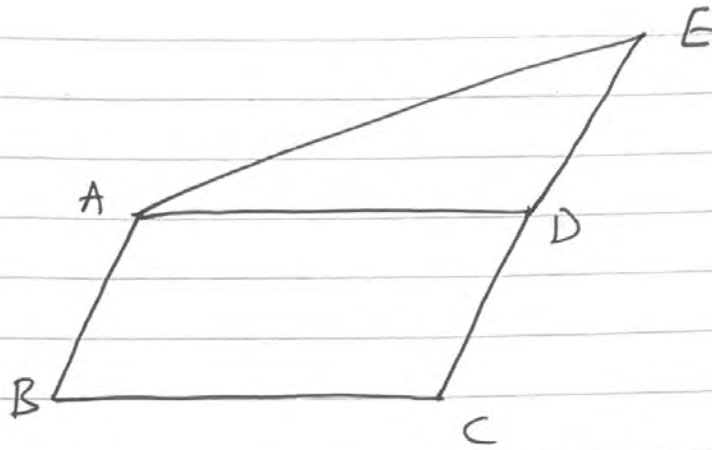
$$\therefore \cos \theta = \frac{16}{6\sqrt{2} \times 4\sqrt{2}} = \frac{1}{3}$$

$$\therefore \theta = \cos^{-1}\left(\frac{1}{3}\right) = 70.5^\circ$$

c) Area $\Delta ABC = \frac{1}{2} \times 4\sqrt{2} \times 6\sqrt{2} \times \sin(70.5^\circ) = 16\sqrt{2}$

$$\therefore \text{Area Parallelogram} = 2 \times 16\sqrt{2} = 32\sqrt{2}$$

d)



$$\triangle ADE = \triangle ABC = \triangle ADC$$

$$\therefore \text{Area } \triangle ADE = 16\sqrt{2} //$$

$$\text{Total area} = 16\sqrt{2} + 32\sqrt{2} = \boxed{48\sqrt{2}}$$