C34 Jan 2014 (MA)

Q1) $y = 2x(x^2+3)^{-1}$

PRODUCT RULE : $\frac{dy}{dx} = -2x(x^2+3)^{-2}(2x) + 2(x^2+3)^{-1}$

$$= \frac{-4x^2}{(x^2+3)^2} + \frac{2}{(x^2+3)}$$

$$\frac{dy}{dx} = \frac{-4x^2 + 2(x^2+3)}{(x^2+3)^2} \quad //$$

$f'(x) > 0$: $-4x^2 + 2x^2 + 6 > 0$

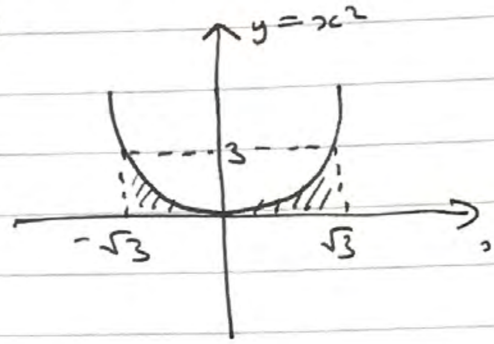
$6 - 2x^2 > 0$

$\frac{6}{2} > x^2$

$x^2 < 3$

$x < \sqrt{3}$

and $x > -\sqrt{3}$



ie $\boxed{-\sqrt{3} < x < \sqrt{3}}$

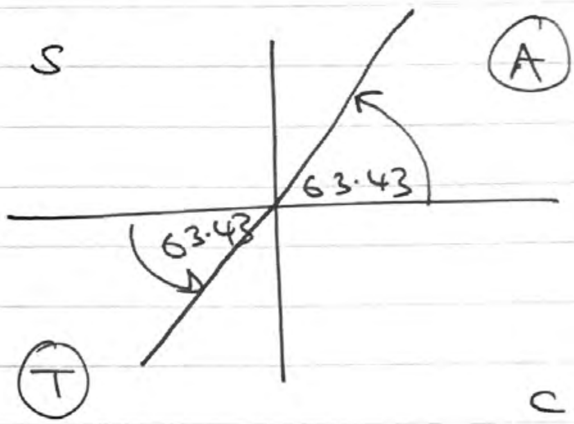
Q2) $\dfrac{\tan 2x + \tan 50}{1 - \tan 2x \tan 50} = 2$ D

LHS = $\tan(2x + 50)$ [addition formulae]

$\Rightarrow \tan(2x + 50) = 2$

$\Rightarrow \tan^{-1}(2) = 2x + 50° = 63.43°$

Solving in: $50 \leq 2x + 50 \leq 590°$



$2x + 50° = 63.43°,$
$180 + 63.43°,$
$360 + 63.43°,$

$2x = 13.43°,$
$193.43°,$
$373.43°,$

$x = \boxed{6.72°, 96.72°, 186.72°}$

Q3a) $4x^3 + 2x^2 + 17x + 8 = Ax^3 + Bx^2 + 4Ax + 4B + Cx + D$

Comparing coefficients :

$x^3 : \{ 4 = A$ //

$x^2 \{ 2 = B$ //

$x \begin{cases} 17 = 4A + C \\ C = 17 - 4(4) = 1 // = C \end{cases}$

constants $\begin{cases} 8 = 4B + D \\ D = 8 - 4(2) = 0 // \end{cases}$

$$\boxed{\begin{aligned} A &= 4 \\ B &= 2 \\ C &= 1 \\ D &= 0 \end{aligned}}$$

b) $\displaystyle\int_1^4 \left[\frac{4x^3 + 2x^2 + 17x + 8}{x^2 + 4}\right] dx$

$= \displaystyle\int_1^4 \left[\frac{(4x+2)(x^2+4) + x}{(x^2+4)}\right] dx$

$= \displaystyle\int_1^4 \left[4x + 2 + \frac{x}{x^2+4}\right] dx$

By pattern !

$= \left[2x^2 + 2x + \frac{1}{2}\ln|x^2+4|\right]_1^4$

$= \left[32 + 8 + \frac{1}{2}\ln(20)\right] - \left[2 + 2 + \frac{1}{2}\ln 5\right]$

$= 40 - 4 + \frac{1}{2}\ln 20 - \frac{1}{2}\ln 5$

$= 36 + \frac{1}{2}\ln 4 = \boxed{36 + \ln 2}$

Q4a) $g(1) = \frac{1+9}{2+3} = \frac{10}{5} = 2$

$fg(1) = f(2) = 2|3-2| + 5 = \boxed{7}$

b) $\underline{x=0}$: $y = \frac{9}{3} = 3$ //

as $x \to \infty$, $g(x) \to \frac{\cancel{\infty}}{2\cancel{\infty}} \left(\frac{1}{2}\right)$ //.

explanation: for very large x, $g(x)$ tends towards $\frac{x}{2x}$ as the $(+9)$ on the top of the fraction and the $(+3)$ on the bottom become very insignificant.
so $g(x) \simeq \frac{x}{2x} \simeq \frac{1}{2}$ //.

Therefore we can conclude that $y = \frac{1}{2}$ is an asymptote.

so range: $\boxed{\frac{1}{2} < g(x) \leq 3}$

c) $y = \frac{x+9}{2x+3}$

$x \leftrightarrow y$; $x = \frac{y+9}{2y+3}$

$x(2y+3) = y+9$
$2xy + 3x - y = 9$
$y(2x-1) = 9-3x$

$\therefore y = \boxed{\frac{9-3x}{2x-1}} = g^{-1}(x)$

domain of $g^{-1}(x)$ = range of $g(x)$

$$\Rightarrow \boxed{\tfrac{1}{2} < x \leq 3}$$

d) for $f(x)$: $x = 0$: $y = 2(3) + 5 = 11$ //.

The lowest point is at $(y=5)$ since the (original) graph is translated in the +ve y-dir. by 5 units.

$$\Rightarrow \boxed{5 < k \leq 11}$$

$y=5$ will have just one root!

(Q 5a) let $y = 2^x$

$\ln(y) = (x)\ln(2)$

$\frac{d}{dx}(\ln y) = \frac{d}{dx}(x\ln 2)$ ~ (IMPLICITLY)

$\frac{1}{y}\frac{dy}{dx} = \ln 2$ //

$\therefore \frac{dy}{dx} = y\ln 2$

$y = 2^x$

$\frac{dy}{dx} = 2^x \ln 2$

$\square$

b) $\frac{d}{dx}\left(2x + 3y^2 + 3x^2y + 12\right) = \frac{d}{dx}\left(4 \cdot 2^x\right)$

$\Rightarrow \quad 2 + 6y\frac{dy}{dx} + 6xy + 3x^2\frac{dy}{dx} = 4 \cdot 2^x \cdot \ln 2$

$\Rightarrow \quad \frac{dy}{dx}\left(6y + 3x^2\right) = 2^x \cdot 4\ln 2 - 2 - 6xy$

$\Rightarrow \quad \frac{dy}{dx} = \frac{2^x(4\ln 2) - 6xy - 2}{3x^2 + 6y}$ //

at P, $\frac{dy}{dx} = \frac{2^2(4\ln 2) - 6(2)(0) - 2}{3(2^2) + 0}$

$= \frac{16\ln 2 - 2}{12} = \frac{8\ln 2 - 1}{6}$ //

$\Rightarrow \quad y - 0 = \frac{8\ln 2 - 1}{6}(x - 2)$

$\Rightarrow \quad y = \frac{(8\ln 2 - 1)}{6}x - \frac{(8\ln 2 - 1)}{3}$

Q6a) $$\frac{6}{\sqrt{9+Ax^2}} = 6(9+Ax^2)^{-\frac{1}{2}} = 6\cdot 9^{-\frac{1}{2}}\left(1+\frac{A}{9}x^2\right)^{-\frac{1}{2}}$$

$$= 2\left(1+\frac{A}{9}x^2\right)^{-\frac{1}{2}} //$$

$$2\left(1+\frac{A}{9}x^2\right)^{-\frac{1}{2}} \simeq 2\left[1-\frac{A}{18}x^2+\frac{-\frac{1}{2}\left(-\frac{3}{2}\right)}{2}\left(\frac{A}{9}x^2\right)^2\right]$$

$$\left[\begin{array}{l} n=-\frac{1}{2} \\ x=\frac{A}{9}x^2 \end{array}\right] \quad \simeq 2-\frac{A}{9}x^2+\frac{2A^2}{216}x^4 //$$

comparing coefficients : $\boxed{B=2}$ //

$$\frac{A}{9} = \frac{2}{3}$$

$$\therefore \boxed{A = \frac{18}{3} = 6} //$$

and $$C = \frac{2A^2}{216}$$

$$C = \frac{2(6^2)}{216} = \boxed{\frac{1}{3}}$$

b) x^4 term : $$2\left[\frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{6}\right]\left(\frac{A}{9}x^2\right)^3$$

$$= 2\left[-\frac{5}{16}\times\frac{A^3}{729}\right]x^6$$

Dont forget the $\times 2$!

$$\therefore \text{coeff.} = 2\left[-\frac{5}{16}\times\frac{(6)^3}{729}\right] = \boxed{\frac{-5}{27}}$$

Q7a) $f(x) = 2x(1+x)\ln x = (2x + 2x^2)\ln x$

PRODUCT RULE $(u = 2x + 2x^2)$, $(v = \ln x)$

$\Rightarrow f'(x) = (2+4x)\ln x + \frac{(2x + 2x^2)}{x}$

$\Rightarrow f'(x) = \boxed{2x + 2 + (2+4x)\ln x}$

b) $f'(x) = 0$: $2x + 2\ln x + 4x\ln x + 2 = 0$

$x + \ln x + 2x\ln x + 1 = 0$

$\ln x(1 + 2x) = -1 - x$

$\therefore \ln x = -\frac{(1+x)}{1+2x}$

$\Rightarrow x = e^{\frac{-(1+x)}{1+2x}}$

$\square$

c) $x_1 = e^{-\frac{1+0.46}{1+2(0.46)}} = \boxed{0.4675}$.

$x_2 = \boxed{0.4684}$. similarly

and $x_3 = \boxed{0.4685}$

d) $x_A = 0.47$ (x_1, x_2, x_3 are all 0.47 to 2 d.p)

$y_A = 2(0.47)(1+0.47)\ln 0.47 \simeq -1.04$

$\therefore A(0.47, -1.04)$

Q8a) $LHS = \dfrac{2}{\sin 2A} - \dfrac{\cos A}{\sin A}$

$= \dfrac{\cancel{2}}{\cancel{2}\sin A\cos A} - \dfrac{\cos A}{\sin A}\dfrac{\times(\cos A)}{\times(\cos A)}$

$= \dfrac{1}{\sin A\cos A} - \dfrac{\cos^2 A}{\sin A\cos A}$

$= \dfrac{1-\cos^2 A}{\sin A\cos A} = \dfrac{\sin^{\cancel{2}} A}{\cancel{\sin A}\cos A}$

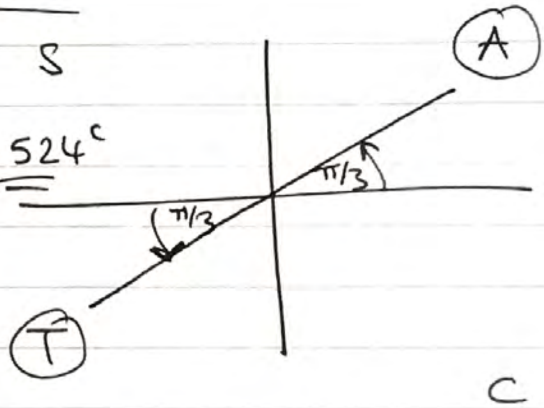
$= \dfrac{\sin A}{\cos A} = \tan A = RHS$ $\square$

bi) $(A = 2\theta)$; $\tan 2\theta = \sqrt{3}$

$\tan^{-1}(\sqrt{3}) = 2\theta = \dfrac{\pi}{3}$

Solving in: $\underline{0 \leq 2\theta \leq \pi}$

$2\theta = \dfrac{\pi}{3}$ $\therefore$ $\boxed{\theta = \dfrac{\pi}{6}} = \underline{\underline{0.524^c}}$



ii) $\tan\theta + \cot\theta = 2\operatorname{cosec}2\theta$ (from a).

$\Rightarrow \dfrac{2}{\sin 2\theta} = 5$

$\Rightarrow \sin 2\theta = \dfrac{2}{5}$ $\therefore 2\theta = \sin^{-1}\left(\dfrac{2}{5}\right)$

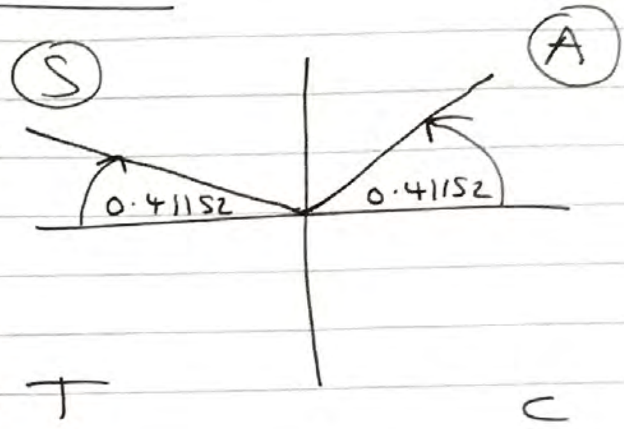
$2\theta = 0.41152$ //

<u>solving in : $0 \leq 2\theta \leq \pi$</u>

$2\theta = 0.4115, \pi - 0.4115$

$2\theta = 0.415, 2.730$

$$\boxed{\theta = 0.206^c, 1.365^c}$$



Q9a) $\int \left[\frac{1}{u} \times (-2\sqrt{x})\right] du$

$u = 4 - \sqrt{x}$

$\frac{du}{dx} = -\frac{1}{2}x^{-\frac{1}{2}}$

$dx = \frac{du}{-\frac{1}{2}x^{-\frac{1}{2}}} = \frac{du\sqrt{x}}{-\frac{1}{2}}$

$dx = -2\sqrt{x}\, du$ //

$= -2\int \left[\frac{\sqrt{x}}{u}\right] du$

but $\sqrt{x} = 4 - u$ //

$\Rightarrow -2\int \left[\frac{4-u}{u}\right] du$

$\Rightarrow -2\int \left[\frac{4}{u} - 1\right] du = -2\left[4\ln|u| - u\right] + c$

$= -8\ln|u| + 2u + c$

$$= \boxed{-8\ln|4-\sqrt{x}| + 8 - 2\sqrt{x} + c}$$

b) $\frac{dh}{dt} > 0$: $4 - \sqrt{h} > 0$

$\sqrt{h} < 4$

$(\sqrt{h})^2 < (4)^2$

$\boxed{0 < h < 16}$ //

$h > 0$ not needed...

c) $\frac{dh}{dt} = \frac{4-\sqrt{h}}{20}$

$\left(\frac{20}{4-\sqrt{h}}\right)\frac{dh}{dt} = 1$

$\Rightarrow 20\int\left(\frac{1}{4-\sqrt{h}}\right)dh = \int (1)\, dt$

$\Rightarrow 20\left[-8\ln|4-\sqrt{h}| + 2(4-\sqrt{h})\right] = t + c$

$\Rightarrow t = 20\left[-8\ln|4-\sqrt{h}| + 2(4-\sqrt{h})\right] - c$

$t=0,\ h=1$: $0 = 20\left[-8\ln 3 + 8 - 2\right] - c$

$\therefore c = 20\left[6 - 8\ln 3\right]$ //

$\Rightarrow t = 20\left[-8\ln|4-\sqrt{h}| + 2(4-\sqrt{h}) - 6 + 8\ln 3\right]$

$h=10$: $t = 20\left[-8\ln|4-\sqrt{10}| + 2(4-\sqrt{10}) - 6 + 8\ln 3\right]$

$t \simeq 117.6\ldots \rightarrow \boxed{118 \text{ years}}$

Q10a) assume intersection

$$\begin{pmatrix} 1+2\lambda \\ 5+\lambda \\ 5-\lambda \end{pmatrix} = \begin{pmatrix} 0+3\mu \\ 2-\mu \\ 12+5\mu \end{pmatrix} \begin{matrix} — (1) \\ — (2) \\ — (3) \end{matrix}$$

(1) : $1+2\lambda = 3\mu$

$\therefore \lambda = \frac{3\mu-1}{2}$

$\hookrightarrow$ (2) : $5 + \frac{3\mu}{2} - \frac{1}{2} = 2-\mu$

$\frac{5}{2}\mu = -\frac{5}{2} \quad \therefore \mu = -1$

sub $\mu = -1$ into (1) : $\lambda = \frac{-3-1}{2} = -2$

sub $\mu = -1$ into (3) : $\lambda = 5 - 12 - 5(-2) = -2$

values of μ / λ are consistent

$\therefore L_1$ and L_2 do ... intersect

$$\text{P.O.I} = \begin{pmatrix} 3(-1) \\ 2--1 \\ 12+5(-1) \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ 7 \end{pmatrix}$$

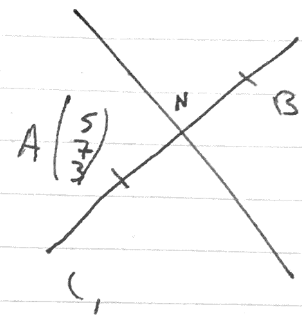
b) using dir. vectors

$$\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} = 6-1-5 = 0$$

$\therefore L_1$ and L_2 are perpendicular.

c) let N be point of intersection. l_2

$\overrightarrow{OB} = \overrightarrow{OA} + 2\overrightarrow{AN}$



$\overrightarrow{ON} = \begin{pmatrix}-3\\3\\7\end{pmatrix}$

$\therefore \overrightarrow{AN} = \overrightarrow{ON} - \overrightarrow{OA} = \begin{pmatrix}-3\\3\\7\end{pmatrix} - \begin{pmatrix}5\\7\\3\end{pmatrix} = \begin{pmatrix}-8\\-4\\4\end{pmatrix}$

$\therefore 2\overrightarrow{AN} = \begin{pmatrix}-16\\-8\\8\end{pmatrix}$

so $\overrightarrow{OB} = \begin{pmatrix}5\\7\\3\end{pmatrix} + \begin{pmatrix}-16\\-8\\8\end{pmatrix} = \boxed{\begin{pmatrix}-11\\-1\\11\end{pmatrix}}$

Q11a) $\underline{y=3}$: $3 = 6\sin t$

$\frac{1}{2} = \sin t$

$t = \sin^{-1}\left(\frac{1}{2}\right) = \boxed{\frac{\pi}{6}}$ (only value within given range of t)

b) $x = 10\cos 2t$ $y = 6\sin t$

$\frac{dx}{dt} = -20\sin 2t$ $\frac{dy}{dt} = 6\cos t$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6\cos t}{-20\sin 2t} = \frac{6\cancel{\cos t}}{-40\sin t\cancel{\cos t}} = \underline{-\frac{3}{20\sin t}}$

b cont.) at $t = \frac{\pi}{6}$, $\frac{dy}{dx} = \frac{-3}{20\sin\frac{\pi}{6}} = -\frac{3}{10}$ //

$\therefore$ at normal, $m = \frac{10}{3}$ // $\left(-\frac{3}{10} \times \frac{10}{3} = -1\right)$

$\therefore$ normal at (5, 3) will have eqn...

$\Rightarrow y - 3 = \frac{10}{3}(x - 5)$

$\Rightarrow y - 3 = \frac{10}{3}x - \frac{50}{3}$

$\Rightarrow y = \frac{10}{3}x - \frac{41}{3}$

$\xrightarrow{\times 3} \boxed{3y = 10x - 41}$ □

c) substituting $(x = 10\cos 2t)$ and $(y = 6\sin t)$,

$18\sin t = 100\cos 2t - 41$

$18\sin t = 100(2\cos^2 t - 1) - 41$

$18\sin t = 100(1 - 2\sin^2 t) - 41$

$18\sin t = 100 - 200\sin^2 t - 41$

$200\sin^2 t + 18\sin t - 59 = 0$ //

By Quadratic Formula... $\sin t = \frac{1}{2}$, $\sin t = -\frac{59}{100}$

(cont) we already know $\sin t = \frac{1}{2}$ at the point A.

so consider $\sin t = -0.59$

$$t = \sin^{-1}(-0.59) = -0.631^c$$

at $t = -0.631^c$, $x = 10\cos(2\times -0.631)$
$= 3.038$
$y = 6\sin(-0.631) = -3.54$

so $\boxed{B(3.04, -3.54)}$

Q12a) $V = \pi \int_0^{\pi/4} y^2 \, dx = \pi \int_0^{\pi/4} x^2 (\sin x + \cos x)^2 \, dx$

$$= \pi \int_0^{\pi/4} x^2(\underbrace{\sin^2 x + \cos^2 x}_{1} + \underbrace{2\sin x \cos x}_{\sin 2x}) \, dx$$

$$= \pi \int_0^{\pi/4} [x^2(1 + \sin 2x)] \, dx$$

$$= \int_0^{\pi/4} [\pi x^2(1+\sin 2x)] \, dx$$

$\square$

b) $V = \pi \int_0^{\pi/4} x^2 + x^2 \sin 2x \ dx$

$= \pi \int_0^{\pi/4} [x^2]\, dx + \pi \int_0^{\pi/4} [x^2 \sin 2x]\, dx$

By parts: $u = x^2 \quad v' = \sin 2x$

$\frac{du}{dx} = 2x \quad v = -\frac{1}{2}\cos 2x$

$= \left[-\frac{1}{2}x^2 \cos 2x\right]_0^{\pi/4} + \frac{1}{2}\int_0^{\pi/4} (2x \cos 2x)\, dx$

$= [0] + \int_0^{\pi/4} [x \cos 2x]\, dx$

By Parts: $u = x \quad v' = \cos 2x$

$\frac{du}{dx} = 1 \quad v = \frac{1}{2}\sin 2x$

$= \left[\frac{1}{2}x \sin 2x\right]_0^{\pi/4} - \frac{1}{2}\int_0^{\pi/4} [\sin 2x]\, dx$

$= \left[\frac{\pi}{8}\right] - [0] - \frac{1}{2}\left[-\frac{1}{2}\cos 2x\right]_0^{\pi/4}$

$= \frac{\pi}{8} - \frac{1}{2}\left[-\frac{1}{2}\cos\frac{\pi}{2} + \frac{1}{2}\cos(0)\right]$

$= \frac{\pi}{8} - \frac{1}{4}$ //

$$\therefore \pi \int_0^{\pi/4} [x^2 \sin 2x]\, dx = \left[\frac{\pi}{8} - \frac{1}{4}\right] \times \pi$$

$$\text{hence} \quad V = \pi \int_0^{\pi/4} [x^2]\, dx + \frac{\pi^2}{8} - \frac{\pi}{4}$$

$$= \pi \left[\frac{x^3}{3}\right]_0^{\pi/4} + \frac{\pi^2}{8} - \frac{\pi}{4}$$

$$= \pi \left[\frac{\pi^3}{192}\right] + \frac{\pi^2}{8} - \frac{\pi}{4}$$

$$= \boxed{\frac{\pi^4}{192} + \frac{\pi^2}{8} - \frac{\pi}{4}}$$