IAL C12 S14

PMT

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1.

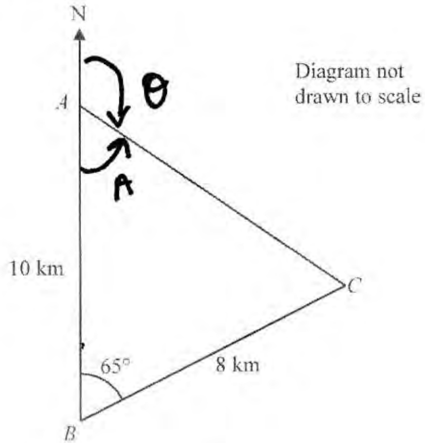


Figure 1

Figure 1 shows the position of three stationary fishing boats A, B and C, which are assumed to be in the same horizontal plane.

Boat A is 10 km due north of boat B.

Boat C is 8 km on a bearing of 065° from boat B.

(a) Find the distance of boat C from boat A, giving your answer to the nearest 10 metres. **(3)**

(b) Find the bearing of boat C from boat A, giving your answer to one decimal place. **(3)**

a) $AC^2 = 8^2 + 10^2 - 2 \times 8 \times 10 \times \cos 65 \quad \therefore AC = 9.81\text{ km}$

b) $\dfrac{\sin A}{8} = \dfrac{\sin 65}{9.81\ldots} \Rightarrow A = 47.6\ldots$

$\therefore \theta = 132.4$

2. Without using your calculator, solve

$$x\sqrt{27} + 21 = \frac{6x}{\sqrt{3}}$$

Write your answer in the form $a\sqrt{b}$ where a and b are integers.

You must show all stages of your working. **(4)**

$x\, 3\sqrt{3} + 21 = 2\sqrt{3}\, x$

$x(3\sqrt{3} - 2\sqrt{3}) = -21$

$x(\sqrt{3}) = -21 \Rightarrow x = \dfrac{-21}{\sqrt{3}} = -7\sqrt{3}$

3. Solve, giving each answer to 3 significant figures, the equations

(a) $4^a = 20$ **(2)**

(b) $3 + 2\log_2 b = \log_2(30b)$ **(5)**

(Solutions based entirely on graphical or numerical methods are not acceptable.)

a) $a \log 4 = \log 20$ ÷ $a = 2.16$

b) $\log_2(30b) - \log_2(b^2) = 3$

$\Rightarrow \log_2\left(\frac{30b}{b^2}\right) = 3 \Rightarrow \frac{30}{b} = 2^3 = 8$

$\Rightarrow 30 = 8b \therefore b = 3.75$

4.

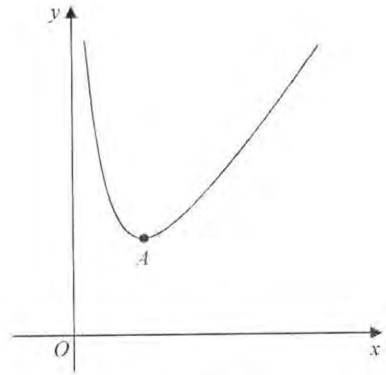


Figure 2

Figure 2 shows a sketch of part of the curve with equation $y = f(x)$ where

$$f(x) = x^2 + \frac{16}{x}, \quad x > 0$$

The curve has a minimum turning point at A.

(a) Find $f'(x)$. **(2)**

(b) Hence find the coordinates of A. **(4)**

(c) Use your answer to part (b) to write down the turning point of the curve with equation

(i) $y = f(x + 1)$.

(ii) $y = \frac{1}{2}f(x)$. **(2)**

a) $f(x) = x^2 + 16x^{-1} \Rightarrow f'(x) = 2x - 16x^{-2}$

b) TP $\Rightarrow f'(x) = 0$ $2x = \frac{16}{x^2} \Rightarrow x^3 = 8 \therefore x = 2$

$y = 4 + \frac{16}{2}$

$A(2, 12)$

c) $f(x+1)$ $\leftarrow$ TP$(1, 12)$ ii) $\frac{1}{2}f(x)$ $\times\frac{1}{2} = \div 2$ TP$(2, 6)$

5.

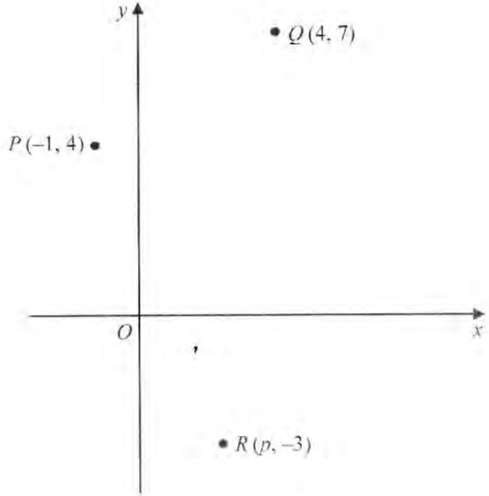


Diagram not drawn to scale

Figure 3

Figure 3 shows the points P, Q and R.

Points P and Q have coordinates $(-1, 4)$ and $(4, 7)$ respectively.

(a) Find an equation for the straight line passing through points P and Q.

Give your answer in the form $ax + by + c = 0$, where a, b and c are integers.

(4)

The point R has coordinates $(p, -3)$, where p is a positive constant.

Given that angle $QPR = 90°$,

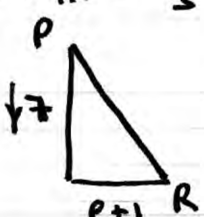
(b) find the value of p.

(3)

a) $M_{PQ} = \frac{3}{5}$ $\quad y - 4 = \frac{3}{5}(x+1) \Rightarrow 5y - 20 = 3x + 3$

$\Rightarrow 3x - 5y + 23 = 0$

b) $M_{PR} = -\frac{5}{3}$ perp to PQ



$\frac{-7}{p+1} = \frac{-5}{3} \Rightarrow 21 = 5p + 5$

$5p = 16$

$p = 3.2$

6. (a) Show that

$$\frac{\cos^2 x - \sin^2 x}{1 - \sin^2 x} \equiv 1 - \tan^2 x, \qquad x \neq (2n+1)\frac{\pi}{2},\ n \in \mathbb{Z}$$

(2)

(b) Hence solve, for $0 \leqslant x < 2\pi$,

$$\frac{\cos^2 x - \sin^2 x}{1 - \sin^2 x} + 2 = 0$$

Give your answers in terms of π.

(5)

a) $\frac{\cos^2 - \sin^2 x}{\cos^2 x} = \frac{\cos^2 x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x} = 1 - \tan^2 x$

b) $1 - \tan^2 x + 2 = 0 \Rightarrow \tan^2 x = 3$

$\Rightarrow \tan x = \sqrt{3}$ $\qquad \tan x = -\sqrt{3}$

$x = \frac{\pi}{3}\ \ \frac{4\pi}{3}$ $\qquad x = -\frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3}$

$+\pi$ $\qquad +\pi \quad +\pi$

$\therefore x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

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7. (i) A curve with equation $y = f(x)$ passes through the point (2, 3).

Given that

$$f'(x) = \frac{4}{x^3} + 2x - 1$$

find the value of f(1).

(5)

(ii) Given that

$$\int_1^4 (3\sqrt{x} + A)\,dx = 21$$

find the exact value of the constant A.

(5)

i) $f'(x) = 4x^{-3} + 2x - 1$

$f(x) = \frac{4x^{-2}}{-2} + \frac{2x^2}{2} - x + c$

$f(x) = -2x^{-2} + x^2 - x + c \qquad (2,3)$

$3 = -\frac{2}{4} + 4 - 2 + c \quad \therefore c = \frac{3}{2}$

$\therefore f(1) = -2(1)^{-2} + (1)^2 - 1 + \frac{3}{2} = \underline{-\frac{1}{2}}$

ii) $\int_1^4 3x^{\frac{1}{2}} + A\,dx = \left[\frac{3x^{\frac{3}{2}}}{\frac{3}{2}} + Ax\right]_1^4$

$= \left[2x^{\frac{3}{2}} + Ax\right]_1^4 = \left[2(8) + 4A - 2 - A\right]$

$\Rightarrow 14 + 3A = 21 \quad \Rightarrow 3A = 7 \quad \Rightarrow A = \underline{\frac{7}{3}}$

8. Given that

$$1 + 12x + 70x^2 + \ldots$$

is the binomial expansion, in ascending powers of x of $(1 + bx)^n$, where $n \in \mathbb{N}$ and b is a constant,

(a) show that $nb = 12$

(1)

(b) find the values of the constants b and n.

(6)

FB $\Rightarrow (1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3$

$(1+bx)^n = 1 + nbx + \frac{n(n-1)}{2}(bx)^2$

$= 1 + [bn]x + \left[\frac{n(n-1)}{2}b^2\right]x^2$

$\therefore bn = 12 \qquad \frac{n(n-1)}{2}b^2 = 70$

$\Rightarrow b = \frac{12}{n} \qquad (n^2 - n)\frac{144}{n^2} = 140$

$\Rightarrow 144n^2 - 144n = 140n^2$

$\Rightarrow 4n^2 - 144n = 0$

$\Rightarrow 4n(n - 36) = 0$

$\therefore \underline{n = 36}$

$b = \frac{12}{36} \Rightarrow \underline{b = \frac{1}{3}}$

blank

9. (i) Find the value of $\sum_{r=1}^{20}(3+5r)$ **(3)**

(ii) Given that $\sum_{r=0}^{\infty}\frac{a}{4^r}=16$, find the value of the constant a. **(4)**

i) $r=1 \quad u_1=8 \qquad \therefore a=8 \qquad$ AS

$r=2 \quad u_2=13 \qquad d=5$

$r=3 \quad u_3=18$

$S_{20}=\frac{1}{2}n[2a+(n-1)d]=10[16+19\times5]$

$=10\times111=\underline{1110}$

$r=0 \quad u_0=a$

ii) $r=1 \quad u_1=\frac{a}{4} \qquad \therefore \underline{GS} \qquad a=a$

$r=2 \quad u_2=\frac{a}{4^2} \qquad r=\frac{1}{4}$

$r=3 \quad u_3=\frac{a}{4^3}$

$\sum_0^\infty \frac{a}{4^r}=a+\frac{a}{4}+\frac{a}{16}+\dots = S_\infty$

$\frac{a}{1-r}=16 \qquad \frac{a}{\frac{3}{4}}=16 \qquad \therefore \underline{a=12}$

10. The equation

$$kx^2+4x+k=2, \text{ where } k \text{ is a constant,}$$

has two distinct real solutions for x.

(a) Show that k satisfies

$$k^2-2k-4<0$$

(4)

(b) Hence find the set of all possible values of k. **(3)**

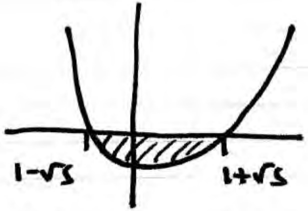
two Solutions $\Rightarrow b^2-4ac>0 \qquad kx^2+4x+(k-2)=0$

$\Rightarrow 16-4k(k-2)>0 \qquad 16-4k^2+8k>0$

$\Rightarrow 0>4k^2-8k-16 \Rightarrow \underline{k^2-2k-4<0}$

b) $k^2-2k-4=0$

$(k-1)^2-1=4 \Rightarrow (k-1)^2=5 \Rightarrow k=1\pm\sqrt5$



$\therefore 1-\sqrt5<k<1+\sqrt5$

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11.

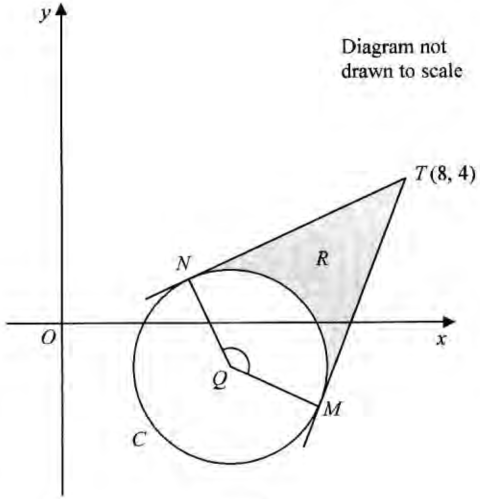


Figure 4

Figure 4 shows a sketch of the circle C with centre Q and equation

$$x^2 + y^2 - 6x + 2y + 5 = 0$$

(a) Find

(i) the coordinates of Q,

(ii) the exact value of the radius of C.

(5)

The tangents to C from the point T (8, 4) meet C at the points M and N, as shown in Figure 4.

(b) Show that the obtuse angle MQN is 2.498 radians to 3 decimal places.

(5)

The region R, shown shaded in Figure 4, is bounded by the tangent TN, the minor arc NM, and the tangent MT.

(c) Find the area of region R.

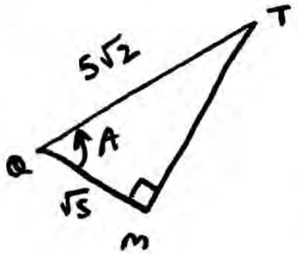
(5)

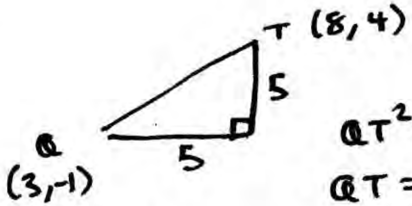
a) $(x-3)^2 - 9 + (y+1)^2 - 1 = -5$

$(x-3)^2 + (y+1)^2 = 5$

Circle $c(3,-1)$ $r = \sqrt{5}$

b)





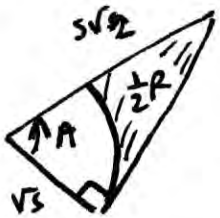
$QT^2 = 5^2 + 5^2$

$QT = 5\sqrt{2}$

$\therefore A = \cos^{-1}\left(\frac{\sqrt{5}}{5\sqrt{2}}\right) = 1.249\ldots$

$\therefore MQN = 2A = 2.49809\ldots \simeq 2.498$

c)



Area of triangle $= \frac{1}{2}(\sqrt{5})(5\sqrt{2}) \sin 1.249\ldots$

$= 7.5$

Area of Sector $= \frac{1}{2}(\sqrt{5})^2(1.249\ldots)$

$= 3.1226\ldots$

$\therefore \frac{1}{2}R = 4.377\ldots$ $\quad \therefore R = 8.75$

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12.

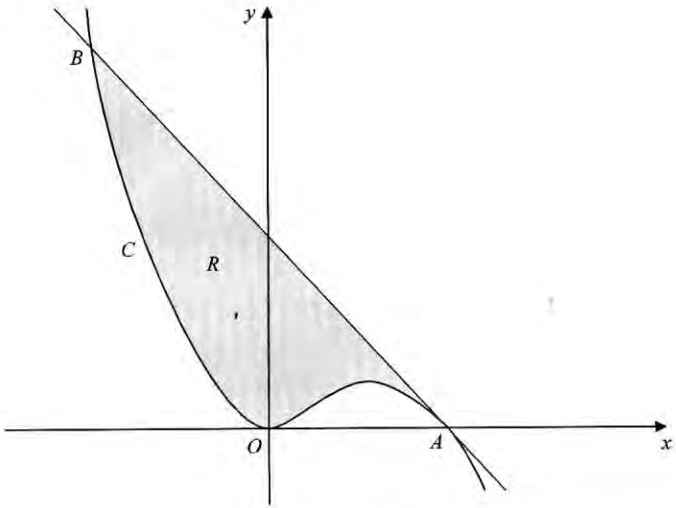


Figure 5

Figure 5 shows a sketch of part of the curve C with equation $y = x^2 - \frac{1}{3}x^3$

C touches the x-axis at the origin and cuts the x-axis at the point A.

(a) Show that the coordinates of A are $(3, 0)$. **(1)**

(b) Show that the equation of the tangent to C at the point A is $y = -3x + 9$ **(5)**

The tangent to C at A meets C again at the point B, as shown in Figure 5.

(c) Use algebra to find the x coordinate of B. **(4)**

The region R, shown shaded in Figure 5, is bounded by the curve C and the tangent to C at A.

(d) Find, by using calculus, the area of region R.

(*Solutions based entirely on graphical or numerical methods are not acceptable.*) **(5)**

a) $y = x^2(1 - \frac{1}{3}x)$ at A $y=0$ $\therefore x^2 = 0,\ \frac{1}{3}x = 1$ $\therefore x = 3$

$A(3,0)$

b) $y' = 2x - x^2$

at A $M_t = 2(3) - 3^2 = -3$ $\Rightarrow y - 0 = -3(x-3)$

$\therefore y = -3x + 9$

c) $-3x + 9 = x^2 - \frac{1}{3}x^3$ $\quad (x-3)$ is a factor

$\Rightarrow \frac{1}{3}x^3 - x^2 - 3x + 9 = 0$

$\Rightarrow x^3 - 3x^2 - 9x + 27 = 0$ $\quad \Rightarrow (x-3)(x^2-9) = 0$

$\times$	x^2	-9	
x	x^3	$-9x$	/
-3	$-3x^2$	$+27$	/

$r = 0$

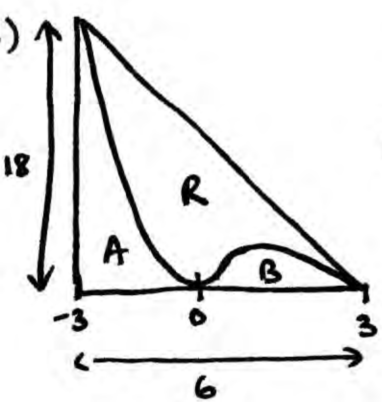
$\therefore x = \cancel{3}, -3$

$x = -3$ $\quad y = (-3)^2(1 - \frac{1}{3}(-3))$

$y = 9 \times 2 = 18$

$\therefore B(-3, 18)$

d)



area of triangle $= \frac{18 \times 6}{2} = 54$

$B + A = \int_{-3}^{3} x^2 - \frac{1}{3}x^3 \, dx = \left[\frac{1}{3}x^3 - \frac{1}{12}x^4\right]_{-3}^{3}$

$= \left[\left(9 - \cancel{\frac{81}{12}}\right) - \left(-9 - \cancel{\frac{81}{12}}\right)\right]$

$= 18$

$\therefore R = 54 - 18 = 36$

13. The height of sea water, h metres, on a harbour wall at time t hours after midnight is given by

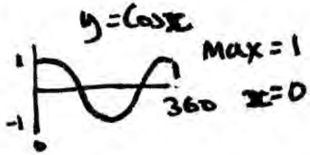
$$h = 3.7 + 2.5\cos(30t - 40)°, \qquad 0 \leqslant t < 24$$

(a) Calculate the maximum value of h and the exact time of day when this maximum first occurs. **(4)**

Fishing boats cannot enter the harbour if h is less than 3

(b) Find the times during the morning between which fishing boats cannot enter the harbour.
Give these times to the nearest minute.

(*Solutions based entirely on graphical or numerical methods are not acceptable.*) **(6)**



max $\cos(30t-40) = 1$ when $30t - 40 = 0$

$\therefore t = \frac{4}{3} \Rightarrow t = 1\text{hr } \frac{1}{3}\text{hr}$

$= 1\text{hr } 20\text{min}$

0120

$\Rightarrow$ max $2.5\cos(30t-40) = 2.5$

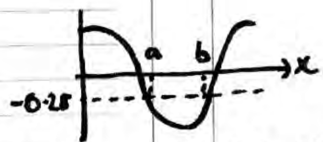
$\times \updownarrow 2.5$

$\therefore$ Max $h = 3.7 + 2.5 = 6.2$m at 0120

b) $h < 3 \Rightarrow 3.7 + 2.5\cos(30t-40) < 3$

$\Rightarrow 2.5\cos(30t-40) < -0.7$

$\cos(30t-40) < -0.28$



$a < x < b \Rightarrow h < 3$

$\cos^{-1}(-0.28) = 106.26°, 253.74...$

360 −

$30t - 40 = 106.26, 253.74$ (+360) 466.26... 613.74..

+40
÷30 $\Rightarrow t = 4.875, 9.79..$ 16.875... 21.79..

$t =$ 0453 0947 1653 2147

No boats between 0453 and 0947 and between 1653 and 2147

14.

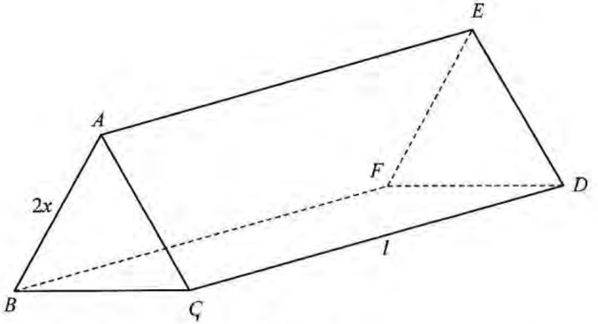


Figure 6

Figure 6 shows a solid triangular prism *ABCDEF* in which $AB = 2x$ cm and $CD = l$ cm.

The cross section *ABC* is an equilateral triangle.

The rectangle *BCDF* is horizontal and the triangles *ABC* and *DEF* are vertical.

The total surface area of the prism is S cm^2 and the volume of the prism is V cm^3.

(a) Show that $S = 2x^2\sqrt{3} + 6xl$

(3)

Given that $S = 960$,

(b) show that $V = 160x\sqrt{3} - x^3$

(5)

(c) Use calculus to find the maximum value of V, giving your answer to the nearest integer.

(5)

(d) Justify that the value of V found in part (c) is a maximum.

(2)

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a) area $\triangle \times 2$ $= \frac{1}{2}(2x)(2x)\sin 60 = 2x^2\frac{\sqrt{3}}{2} = x^2\sqrt{3}$

$A = 2xl$ $\quad \times 3$ $\quad \therefore S = 2x^2\sqrt{3} + 3\times 2xl$

$\therefore S = 2x^2\sqrt{3} + 6xl$ #

b) $S = 2\sqrt{3}x^2 + 6xl = 960$ $\qquad V = \triangle \times l$

$\Rightarrow 6xl = 960 - 2\sqrt{3}x^2$ $\qquad V = x^2\sqrt{3}\times l.$

$\div 6x$ $\quad l = \frac{160}{x} - \frac{\sqrt{3}}{3}x$ $\qquad \Rightarrow V = x^2\sqrt{3}\left(\frac{160}{x} - \frac{\sqrt{3}}{3}x\right)$

$\therefore V = 160\sqrt{3}x - x^3$ #

c/d) $V = 160\sqrt{3}x - x^3$ $\qquad$ max V when $V' = 0$

$V' = 160\sqrt{3} - 3x^2$ $\qquad 160\sqrt{3} = 3x^2 \Rightarrow x = \sqrt{\frac{160\sqrt{3}}{3}}$

$V'' = -6x$ $\qquad x = 9.611\ldots$ at max V

$\therefore$ max $V = 160\sqrt{3}(9.611..) - 9.611..^3$

$\therefore$ max $V = 1776$ cm^3

at V $x = 9.611..$

$V'' = -57.7$ $\quad \therefore V'' < 0$ $\quad \therefore$ V is at a maximum