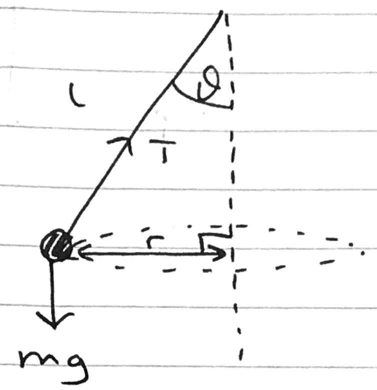
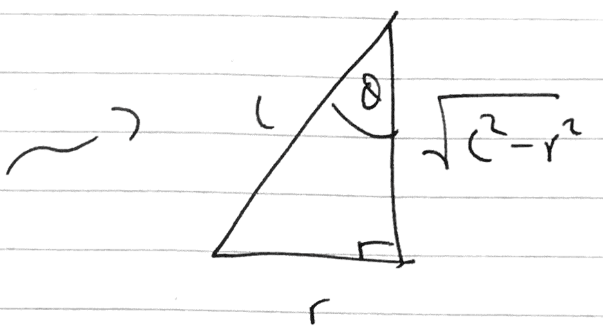
M3 June 2016 IAL (MA)

Q1)





$\therefore \cos\theta = \frac{\sqrt{l^2-r^2}}{l}$

$R(\updownarrow)$: $T\cos\theta = mg$ — ① & $\tan\theta = \frac{r}{\sqrt{l^2-r^2}}$

$\xrightarrow{\quad\quad} +$

<u>N2L (particle)</u> : $T\sin\theta = mr\omega^2$ — ②

$\frac{②}{①}$: $\frac{\cancel{T}\sin\theta}{\cancel{T}\cos\theta} = \frac{\cancel{m}r\omega^2}{\cancel{m}g}$

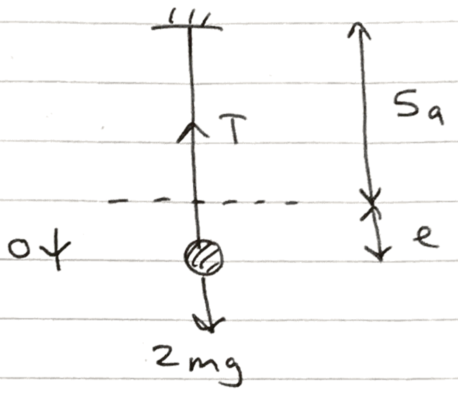
$\tan\theta = \frac{r\omega^2}{g}$

but $\tan\theta = \frac{r}{\sqrt{l^2-r^2}}$

$\therefore \frac{\cancel{r}}{\sqrt{l^2-r^2}} = \frac{\cancel{r}\omega^2}{g}$

<u>×g</u> : $\omega^2 = \frac{g}{\sqrt{l^2-r^2}}$

Q2a) $R(\updownarrow)$: $T = 2mg$



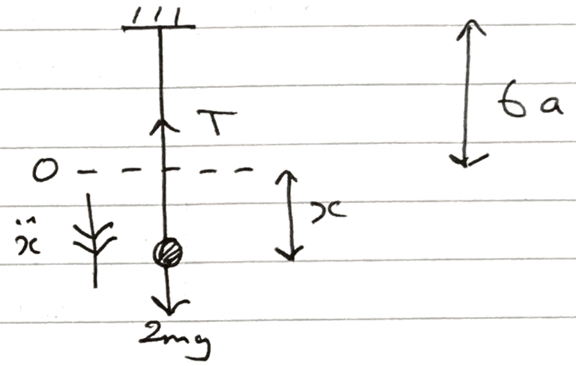
$T = \frac{\lambda x}{l} = \frac{10mg}{5a}(e)$

$\therefore \frac{2mge}{a} = 2mg$

$\div 2mg$: $\frac{e}{a} = 1 \quad \therefore e = a$

so $AO = 5a + a = \boxed{6a}$

b) $\downarrow^{+}$ N2L(P): $2mg - T = 2m\ddot{x}$



$T = \frac{\lambda x}{l} = \frac{2mg}{a}(6a + x - 5a)$

$T = \frac{2mg}{a}(a + x)$

$\therefore 2mg - \frac{2mg}{a}(a+x) = 2m\ddot{x}$

$\cancel{2mg} - \cancel{2mg} - \frac{2mgx}{a} = 2m\ddot{x}$

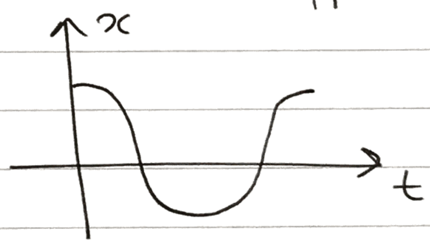
$-\frac{\cancel{2m}g}{a}x = \cancel{2m}\ddot{x}$

$\therefore \ddot{x} = -\frac{g}{a}x$

hence P moves with S.H.M

finding where $\ddot{x}$ is positive

P starts at an endpoint so $x = a\cos\omega t$ applies



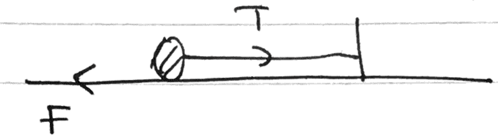
as can be seen from the graph x is max at $t=0$. So at the point of release P is at max displacement. This means x is increasing in the direction AO so $\ddot{x}$ is also positive in this direction

c) $\ddot{x} = -\frac{g}{a}x$

so $\omega = \sqrt{\frac{g}{a}}$

$T = 2\pi \times \frac{1}{\omega} = \boxed{2\pi\sqrt{\frac{a}{g}}}$

Q3a)



at $OB = l$, string is slack.

Initially : $EPE = \frac{4mg}{2l}\left(\frac{l}{4}\right)^2$

$KE = 0$

Finally : $EPE = 0$

[when P moves $\frac{l}{4}$ metres] $KE = ?$

W.D due to friction $= \mu Rs$

$= \frac{2}{5}mg\left(\frac{l}{4}\right)$

so $\frac{2mg}{l}\left(\frac{l}{4}\right)^2 = KE + \frac{2mgl}{20}$

$KE = \frac{2mgl^2}{16l} - \frac{2mgl}{20}$

$KE = \frac{2mgl}{16} - \frac{2mgl}{20} = \frac{mgl}{40} > 0$

$KE > 0$ when string is slack so $OB < l$

b) At A: $EPE = \frac{mgl}{8}$ (from a)

$KE = 0$

when P is at B: $EPE = 0$ (string is slack)

$KE = 0$

W.D due to friction $= \frac{2}{5}mgd$

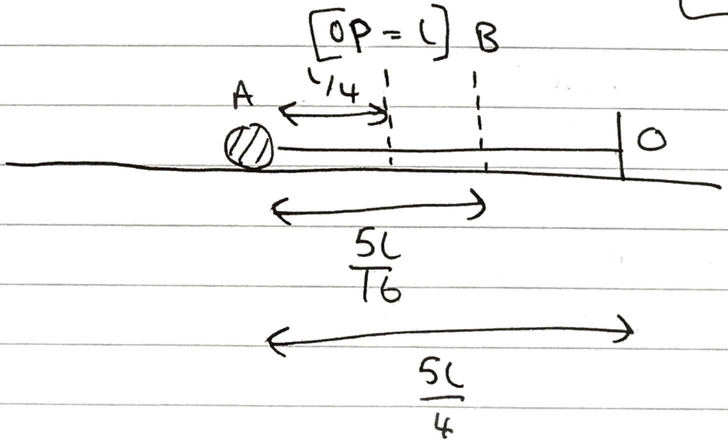
(let d = distance AB)

so $\underbrace{\frac{mgl}{8}}_{\text{Total energy at A}} = \underbrace{\frac{2}{5}mgd}_{\text{work done by friction}}$

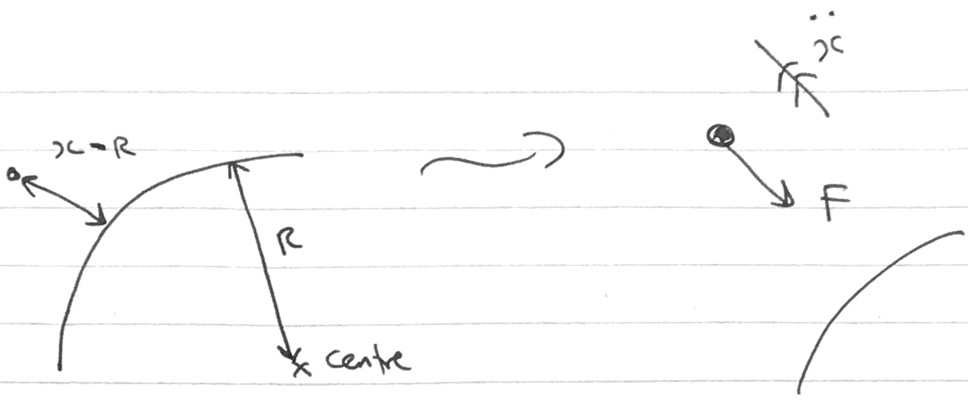
$\Rightarrow \frac{l}{8} = \frac{2}{5}d$

$\Rightarrow d = \frac{5l}{16} = AB$

$\therefore OB = \frac{5}{4}l - \frac{5l}{16} = \boxed{\frac{15l}{16}}$



Q4a)



$$F = \frac{k}{x^2} \quad \text{where } k \text{ is a constant.}$$

at surface of earth, $x = R$, $F = mg$:

$$mg = \frac{k}{R^2}$$

$$\therefore k = mgR^2$$

So $F = \dfrac{mgR^2}{x^2}$

b) greatest height above surface $= \dfrac{R}{20}$ means that at $x = \left(R + \dfrac{R}{20}\right)$, $V = 0$.

$$F = \frac{-\cancel{m}gR^2}{x^2} = \cancel{m}a$$

$$-\frac{gR^2}{x^2} = V\frac{dv}{dx}$$

$$\int (v)\,dv = -gR^2 \int (x^{-2})\,dx$$

$$\frac{v^2}{2} = -gR^2\left[-\frac{1}{x}\right] + c$$

$$\frac{v^2}{2} = \frac{gR^2}{x} + c$$

$x = \frac{21R}{20}$, $v = 0$: $0 = \frac{gR^2}{\frac{21R}{20}} + c$

$$\therefore c = -\frac{20gR}{21}$$

$$\therefore \frac{v^2}{2} = \frac{gR^2}{x} - \frac{20gR}{21}$$

at $x = R$, $v = u$: $\frac{u^2}{2} = \frac{gR^2}{R} - \frac{20gR}{21}$

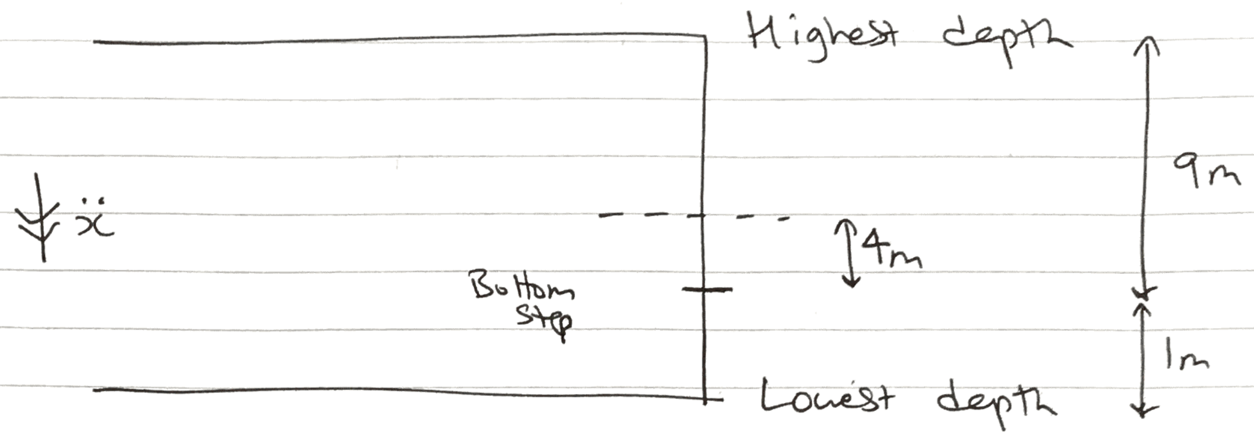
$$\frac{u^2}{2} = gR - \frac{20}{21}gR$$

$$\frac{u^2}{2} = \frac{1}{21}gR$$

$$u^2 = \frac{2}{21}gR$$

$$\therefore u = \sqrt{\frac{2gR}{21}}$$

Q5a)



$T = 12 \text{ hours } 15\text{min} = \frac{49}{4} \text{ hours}$

$\omega = \frac{2\pi}{T} = \frac{8\pi}{49} \text{ rad/hour}$

$a = \frac{9+1}{2} = 5\text{m}$ //

$v^2 = (a^2 - x^2)\omega^2$

at bottom step, $x = 4$

$v^2 = (5^2 - 4^2)\left(\frac{8\pi}{49}\right)^2 = 2.3677\ldots$

$\therefore \boxed{v = 1.54 \text{ m/s}}$

b) $x = 5\cos\left(\frac{8\pi t}{49}\right)$

at $x = 4$: $\frac{4}{5} = \cos\left(\frac{8\pi t}{49}\right)$

$t = \frac{49}{8\pi}\cos^{-1}\left(\frac{4}{5}\right)$ (= time to bottom step)

So total time in one oscillation where ladder is in the water $= 12.25 - 2\left(\frac{49}{8\pi}\right)\cos^{-1}\left(\frac{4}{5}\right)$

$= \boxed{9.7 \text{ hours}}$

x2 as bottom step will re-emerge from the water near the end of the oscillation.

Q6a) $\underline{\Delta GPE = \Delta KE}$

$$0.2g(0.1) = \frac{0.2}{2}(v^2 - u^2)$$

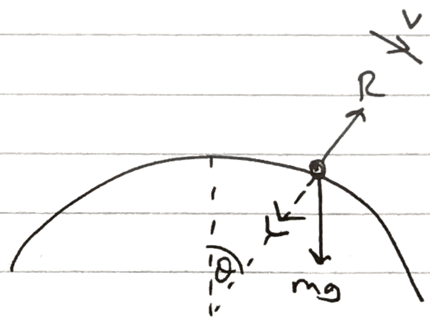
KE is higher at B as its a 'lower' point than A $\therefore v > u$.

$\underline{\times 10}$: $0.2g = v^2 - u^2$

$$v^2 = u^2 + 0.2g$$

$$v^2 = u^2 + 1.96$$

b) $\underline{\text{at } B}$:



$+\swarrow$ $\underline{N2L(P)}$: $mg\cos\theta - R = \frac{mv^2}{r}$

$$0.2g\cos\theta - R = \frac{0.2v^2}{0.5}$$

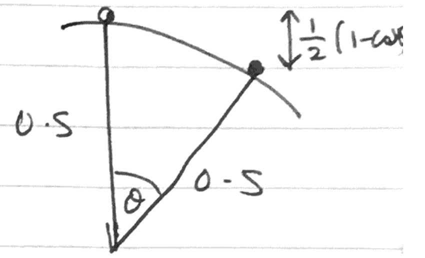
$\underline{R = 0 \text{ as P loses contact}}$: $\cancel{0.}2g\cos\theta = \frac{\cancel{0.}2v^2}{0.5}$

$\Rightarrow g\cos\theta = 2v^2 \quad \therefore v^2 = \frac{g\cos\theta}{2}$.

but B is 0.1m vertically below A.

so $0.5 - 0.5\cos\theta = 0.1$

$0.5(1-\cos\theta) = 0.1$



$1-\cos\theta = \frac{1}{5}$

$\therefore \cos\theta = \frac{4}{5}$

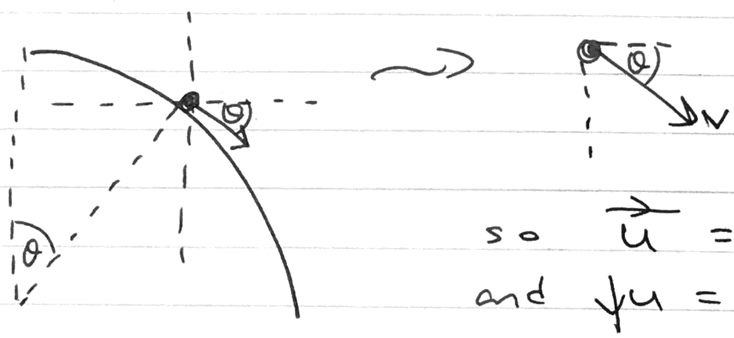
so $v^2 = \frac{g}{2}\left(\frac{4}{5}\right) = \frac{4g}{10} = \frac{2g}{5}$ //

and $v^2 = u^2 + 1.96$

$u^2 = v^2 - 1.96$

$u^2 = \frac{2g}{5} - 1.96$

$u = \sqrt{\frac{2g}{5} - 1.96} = \boxed{1.40}$

c) 

so $\overrightarrow{u} = v\cos\theta$

and $\downarrow u = v\sin\theta$

Suvat from B to ground :

$$\left.\begin{aligned} s &= 0.5\left(\tfrac{4}{5}\right) \\ u &= v\sin\theta = \tfrac{3}{5}\sqrt{\tfrac{2g}{5}} \\ v &= \\ a &= g \\ t &= t \end{aligned}\right\} \quad \begin{aligned} s &= ut + \tfrac{1}{2}at^2 \\ \tfrac{2}{5} &= \tfrac{3}{5}t\sqrt{\tfrac{2g}{5}} + \tfrac{g}{2}t^2 \\ 4.9t^2 &+ \tfrac{21\sqrt{2}}{25} - 0.4 = 0 \end{aligned}$$

$\downarrow +$

By Quadratic formula, $t = 0.18915\ldots$

reject other solution as $t > 0$.

now consider horizontal motion,

$\vec{s} = ut$

$\vec{s} = \frac{4}{5}\sqrt{\frac{2g}{5}}\, t$

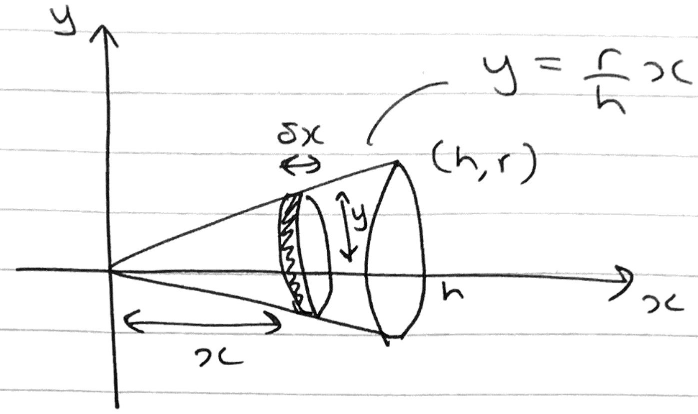
$\vec{s} = \frac{4}{5}\sqrt{\frac{2g}{5}} \times 0.18915\ldots = 0.299598\ldots$

$\simeq 0.3\text{m}$

so $OC = OB\sin\theta + 0.3$

$= 0.5 \times \frac{3}{5} + 0.3 = \boxed{0.60\text{m}}$

Q7a)



splitting up the cone into an infinite amount of 'thin' discs each of thickness δx.

mass of entire cone = $m = p \times$ volume

$$m = p \times \frac{1}{3}\pi r^2 h$$

mass of one disc = $\delta m = p \times [\pi y^2 \delta x]$

distance of c.o.m of one disc from O = x.

recall from M2 that $\bar{x}\sum m_i = \sum m_i x_i$

$$\Rightarrow \bar{x}\left(\frac{p\pi r^2 h}{3}\right) = \sum_{x=0}^{h}\left[p\pi y^2 x\,\delta x\right]$$

$$\lim_{\delta x \to 0} \sum_{x=0}^{h} p\pi y^2 x\,\delta x = p\pi \int_0^h (y^2 x)\,dx$$

$$p\pi \int_0^h (y^2 x)\,dx = p\pi \int_0^h \left(\frac{r^2 x^3}{h^2}\right) dx$$

$$= \frac{p\pi r^2}{h^2}\left[\frac{x^4}{4}\right]_0^h = \frac{p\pi r^2 h^2}{4}$$

so $\frac{p\pi r^2 h^2}{4} = \left(\frac{ph\pi r^2}{3}\right)\bar{x}$

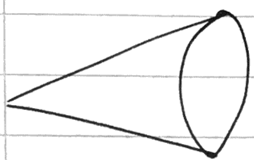
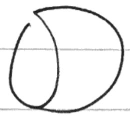
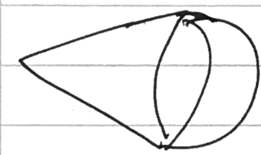
$\div p\pi r^2 h$

$\Rightarrow \frac{h}{4} = \frac{\bar{x}}{3}$

$\times 3$

$\Rightarrow \bar{x} = \frac{3h}{4}$

b)

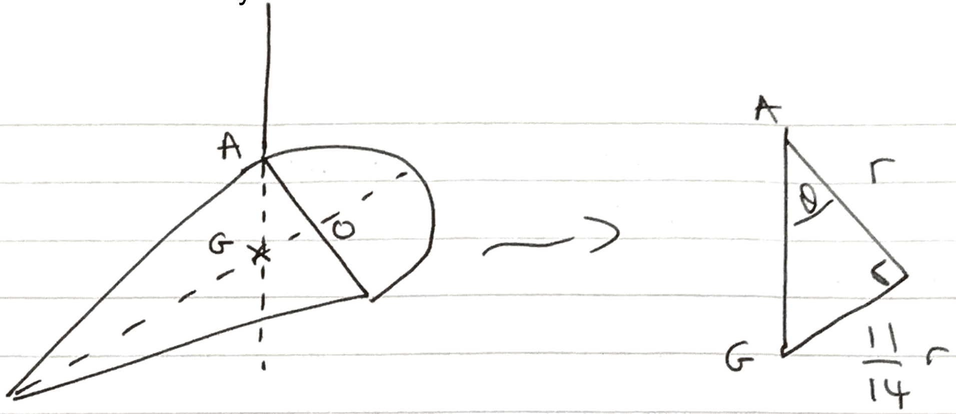
	Shape	Mass (vol.)	Distance of c.o.m from O
+		$\boxed{\frac{\pi r^2(5r)}{3}}$	$5r - \frac{3(5r)}{4} = \boxed{\frac{5r}{4}}$
+		$\boxed{\frac{2}{3}\pi r^3}$	$\boxed{-\frac{3r}{8}}$
=		$\boxed{\frac{7}{3}\pi r^3}$	$\boxed{\bar{x}}$

Moments about O . . .

$$\frac{5}{3}\left(\frac{5r}{4}\right) + \left(\frac{2}{3}\right)\left(-\frac{3}{8}r\right) = \frac{7}{3}(\bar{x})$$

$$\bar{x} = \frac{\frac{25r}{12} - \frac{6r}{24}}{\frac{7}{3}} = \boxed{\frac{11r}{14}}$$

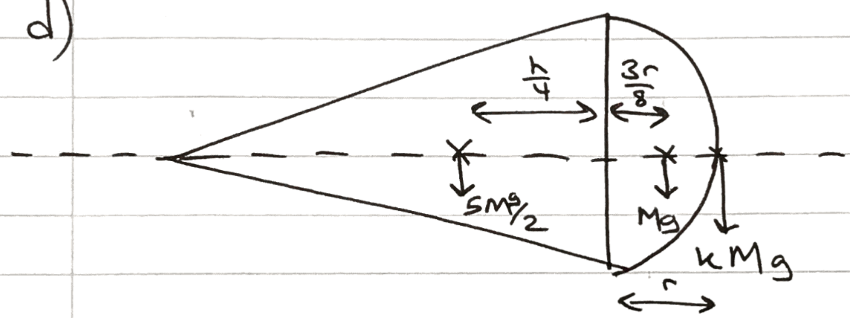
c)



$$\tan\theta = \frac{\frac{11}{14}\cancel{r}}{1\cancel{r}} = \frac{11}{14}$$

$$\theta = \tan^{-1}\frac{11}{14} = \boxed{38.2^\circ}$$

d)



from b, mass of cone $= \frac{5}{2} \times$ mass of hemisphere

so $M_{cone} = \frac{5}{2}M$ //

<u>Moments about O</u> : $\frac{5Mg}{2}\left(\frac{h}{4}\right) = kMg(r) + Mg\left(\frac{3r}{8}\right)$

<u>$\div Mg$</u> : $\frac{5h}{8} = kr + \frac{3r}{8}$

<u>$h = 5r$</u> : $\frac{25r}{8} - \frac{3r}{8} = kr$

$\therefore k = \frac{25}{8} - \frac{3}{8} = \boxed{\frac{11}{4}}$ //