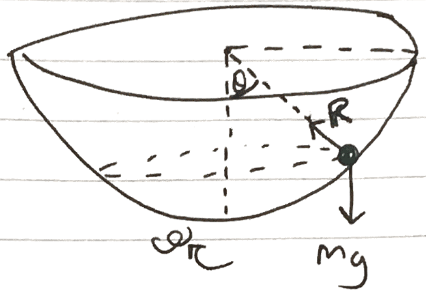
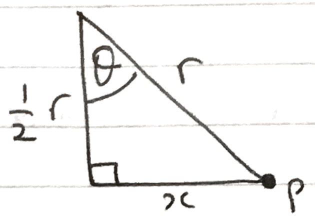
M3 June 2015 IAL (MA)

Q1)





$\sqrt{r^2-(\frac{1}{2}r)^2} = \frac{r\sqrt{3}}{2} = x$

$\therefore \cos\theta = \frac{1}{2}$

$\sin\theta = \frac{\sqrt{3}}{2}$

$R(\updownarrow) \quad R\cos\theta = mg$

$\frac{R}{2} = mg$

$\therefore R = 2mg$

$\overset{+}{\leftarrow}$ N2L (P): $R\sin\theta = mx\omega^2$

$R\left(\frac{\sqrt{3}}{2}\right) = mr\frac{\sqrt{3}}{2}\omega^2$

$R = mr\omega^2.$

so $2\cancel{m}g = \cancel{m}r\omega^2.$

$\omega^2 = \frac{2g}{r} \quad \therefore \omega = \sqrt{\frac{2g}{r}}.$

$T = 2\pi \times \frac{1}{\omega} = 2\pi\sqrt{\frac{r}{2g}}$

$2\pi\sqrt{\frac{r}{2g}} = \pi\sqrt{\frac{4r}{2g}} = \pi\sqrt{\frac{2r}{g}}$

Q2a) $F = \frac{k}{x^2}$

at the surface of the earth, $\underline{x = R \text{ and } F = mg}$

$\therefore \quad mg = \frac{k}{R^2}$

$\therefore k = \underline{\underline{mgR^2}}$

b) $F = -\frac{mgR^2}{x^2}$ (F is negative as its directed towards the centre of the earth)

$-\frac{mgR^2}{x^2} = ma$

$-\frac{gR^2}{x^2} = v\frac{dv}{dx}$

$\int (v)\,dv = -gR^2 \int (x^{-2})\,dx$

$\frac{v^2}{2} = -gR^2\left(-\frac{1}{x}\right) + c$

$\frac{v^2}{2} = \frac{gR^2}{x} + c$

$\underline{x = 3R,\ v = V}$: $\frac{V^2}{2} = \frac{gR^2}{3R} + c$

$\frac{V^2}{2} - \frac{gR}{3} = c$ //

$$\therefore \frac{v^2}{2} = \frac{gR^2}{x} + \frac{V^2}{2} - \frac{gR}{3}.$$

$$\underline{x = R} : \frac{v^2}{2} = gR + \frac{V^2}{2} - \frac{gR}{3}$$

$$v^2 = \frac{4gR}{3} + V^2$$

$$\therefore v = \boxed{\sqrt{V^2 + \frac{4gR}{3}}}$$

Q3a) $a = -2(t+4)^{-\frac{1}{2}}$

$$\frac{dv}{dt} = a = -2(t+4)^{-\frac{1}{2}}$$

$$\therefore \int (1)\,dv = -2\int (t+4)^{-\frac{1}{2}}\,dt$$

$$v = -2\left[2(t+4)^{\frac{1}{2}}\right] + c$$

$$v = -4(t+4)^{\frac{1}{2}} + c$$

$$\underline{t=0,\ v=8} : 8 = -4(2) + c$$

$$c = 16$$

$$\therefore v = -4(t+4)^{\frac{1}{2}} + 16$$

b) $\underline{v=0}$: $16 - 4\sqrt{t+4} = 0$

$$\sqrt{t+4} = 4$$

$$t+4 = (4)^2$$

$$t+4 = 16$$

$$\underline{\underline{t = 12}}.$$

$$v = 16 - 4(t+4)^{\frac{1}{2}}$$

$$x = \int v \, dt$$

$$x = \int \left[16 - 4(t+4)^{\frac{1}{2}}\right] dt$$

$$x = 16t - \frac{4(t+4)^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$x = 16t - \frac{8}{3}(t+4)^{\frac{3}{2}} + c$$

$\underline{t=0, x=0}$: $0 = -\frac{8}{3}(4)^{3/2} + c$

$$c = \frac{64}{3}$$

$$\therefore \; x = 16t - \frac{8}{3}(t+4)^{3/2} + \frac{64}{3}.$$

at $\underline{t=12}$: $x = 16(12) - \frac{8}{3}(16)^{3/2} + \frac{64}{3} = \boxed{\frac{128}{3}}$ m

Q4a) particle moves in complete circles $\therefore T \geq 0$ at top.

<u>Energy from A to highest point:</u>

<u>At A</u> : $KE = \frac{1}{2}(3m)u^2$
$GPE = 0$

<u>At top</u> : $KE = \frac{1}{2}(3m)v^2$
$GPE = 3mga$

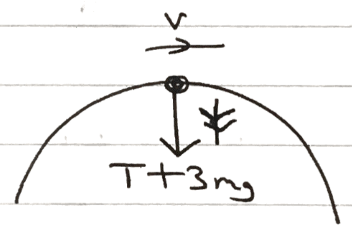
<u>C.O.E</u> : $\frac{\cancel{3}\cancel{m}u^2}{2} = \frac{\cancel{3}\cancel{m}v^2}{2} + \cancel{3}a\cancel{m}g$

$$\frac{u^2}{2} = \frac{v^2}{2} + ag$$

$$u^2 = v^2 + 2ag$$

$$\therefore v^2 = u^2 - 2ag$$

<u>At top</u> :



$\downarrow^{+}$ <u>N2L(particle)</u>: $T + 3mg = \frac{(3m)(v)^2}{a}$

$$T = \frac{3m}{a}(u^2 - 2ag) - 3mg$$

$$T = \frac{3mu^2}{a} - 6mg - 3mg$$

$$T = \frac{3mu^2}{a} - 9mg$$

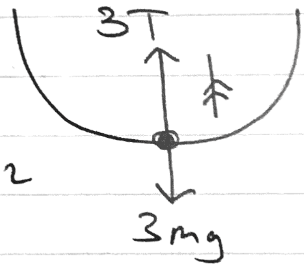
$T \geq 0$ at top : $\frac{3\cancel{m}u^2}{a} - 9\cancel{m}g \geq 0$

$$\frac{3u^2}{a} \geq 9g$$

$\underline{\div 3}$: $\quad u^2 \geq 3ag$

b) at top, T is minimum. $\left(T = \frac{3mu^2}{a} - 9mg\right)$

at bottom, T is max.



$\overset{+}{\uparrow}$ $\underline{\text{N2L}}$ (particle) : $3T - 3mg = \frac{3m}{a}v^2$

<u>Energy from A to bottom to find v :</u>

$$\underbrace{\frac{3}{2}mv^2 - \frac{3}{2}mu^2}_{\Delta KE} = \underbrace{3mga}_{\Delta GPE} \quad (\Delta GPE = \Delta KE)$$

$$\therefore \frac{v^2}{2} - \frac{u^2}{2} = ag$$

$$v^2 = u^2 + 2ag$$

so : $3T - 3mg = \frac{3m}{a}(u^2 + 2ag)$

$$3T - 3mg = \frac{3mu^2}{a} + 6mg$$

$$3T = \frac{3mu^2}{a} + 9mg \quad \text{— } ②$$

$$//$$

and $\quad T = \frac{3mv^2}{a} - 9mg$

$$\times 3 \; \Downarrow$$

$$3T = \frac{9mu^2}{a} - 27mg \quad \text{— } ①$$

equate ① and ② : $\frac{9mu^2}{a} - 27mg = \frac{3mu^2}{a} + 9mg$

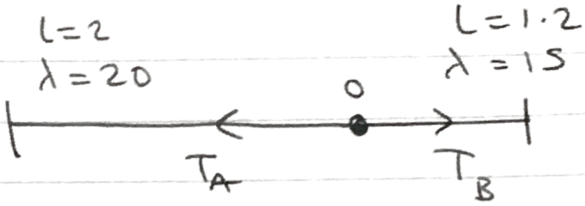
$$\Rightarrow \quad \frac{6\cancel{m}u^2}{a} = 36\cancel{m}g$$

$$\Rightarrow \quad 6u^2 = 36ag$$

$\div 6$

$$\Rightarrow \quad u^2 = 6ag$$

Q5a)



$$T_A = T_B$$

$$\frac{20}{2}(AO-2) = \frac{15}{1.2}(5-AO-1.2)$$

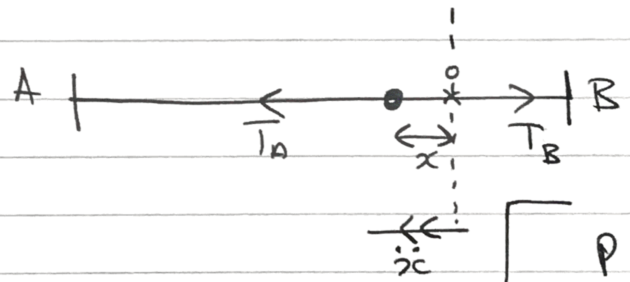
$$10(AO-2) = \frac{25}{2}(3.8-AO)$$

$$10AO - 20 = 47.5 - 12.5AO$$

$$22.5AO = 67.5$$

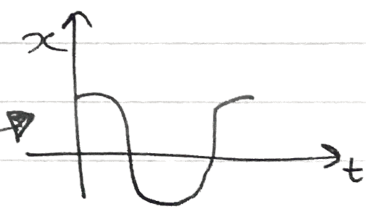
$$\therefore AO = \frac{67.5}{22.5} = \boxed{3\text{m}}$$

b)



why $\ddot{x}$ is positive:

P starts at an endpoint. (at $t=0$). so $x = a\cos\omega t$ applies.



this means x is maximum at $t=0$ as can be seen from the graph, so x is increasing in the direction OC hence $\ddot{x}$ is also increasing in this direction. ($\ddot{x}$ is +ve in direction OC.)

$+\leftarrow$

$$\underline{\text{N2L}}(P): T_A - T_B = 0.5\ddot{x}$$

$$T_A = \frac{20}{2}\left((3-x)-2\right) = \underline{\underline{10(1-x)}}$$

$$T_B = \frac{15}{1.2}(5-3+x-1.2)$$

$$\therefore T_B = \underline{\underline{\frac{25}{2}(0.8+x)}}$$

$\Rightarrow 10(1-x) - 12.5(0.8+x) = 0.5\ddot{x}$

$\Rightarrow 10 - 10x - 10 - 12.5x = 0.5\ddot{x}$

$\Rightarrow -22.5x = \frac{1}{2}\ddot{x}$

$\times 2$

$\Rightarrow -45x = \ddot{x}$ $\therefore$ S.H.M

c) $OB = 2m.$ remember x is positive in direction OC.

so $(x = -0.8)$ when PB is slack.

$x = a\cos\omega t$

$x = 1\cos(t\sqrt{45})$

$-0.8 = \cos(t\sqrt{45})$

$\cos^{-1}(-0.8) = t\sqrt{45} = 2.498...$

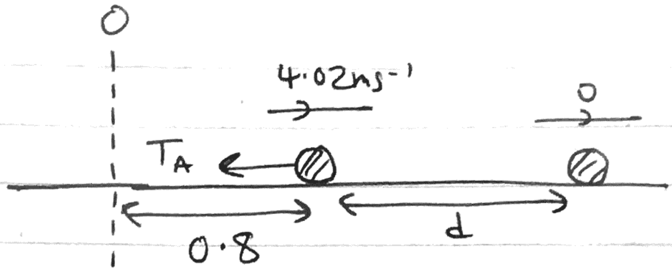
$t = \frac{2.498...}{\sqrt{45}} = 0.3724.$

$\dot{x} = -\sqrt{45}\sin(t\sqrt{45})$

$t = 0.3724..$: $\dot{x} = -\sqrt{45}\left(\frac{3}{5}\right)$

$\therefore$ speed $= \boxed{4.02\,ms^{-1}}$

d)



P is now under the influence of just T_A as PB is slack.

Consider energy:

Initially: $KE = \frac{1}{2}(0.5)\left(\frac{9}{25}\right)(45) = \frac{81}{20}$

$EPE = \frac{20}{4}(3+0.8-2)^2 = \frac{81}{5}$

At rest: $KE = 0$

$EPE = \frac{20}{4}(3+0.8+d-2)^2$

C.O.E: $\frac{81}{20} + \frac{81}{5} = 5(1.8+d)^2$

$\frac{81}{4} = 5(d+1.8)^2$

$(d+1.8)^2 = \frac{81}{20}$

$d + 1.8 = \pm\sqrt{\frac{81}{20}}$

$d = -1.8 \pm \sqrt{\frac{81}{20}}$

$d > 0 \quad \therefore \quad d = -1.8 + \sqrt{\frac{81}{20}}$

so $AD = 3.8 + d = 4.012\ldots$

$\therefore BD = 5 - 4.012\ldots = \boxed{0.99\text{m}}$

Q6a) $V = \pi\int_0^2 y^2\,dx = \pi\int_0^2 (x^2+3)^2\,dx$

$$= \pi\int_0^2 (x^4 + 6x^2 + 9)\,dx$$

$$= \pi\left[\frac{x^5}{5} + \frac{6x^3}{3} + 9x\right]_0^2 = \pi\left[\frac{202}{5}\right] - \pi\left[0\right]$$

$$= \frac{202\pi}{5}$$

b) $M\bar{x} = \int_0^2 \pi y^2 x\,dx = \pi\int_0^2 x(x^2+3)^2\,dx$

By pattern.

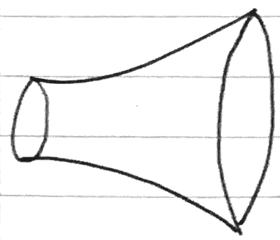
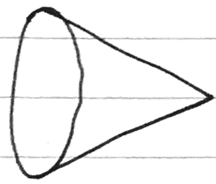
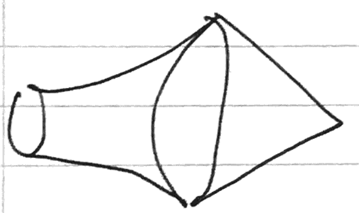
$$= \pi\left[\frac{\frac{1}{2}(x^2+3)^3}{3}\right]_0^2 = \pi\left[\frac{1}{6}(x^2+3)^3\right]_0^2$$

$$= \pi\left[\frac{1}{6}(7)^3\right] - \pi\left[\frac{1}{6}(3)^3\right]$$

$$= \frac{\pi}{6}\left[7^3 - 3^3\right] = \frac{158\pi}{3}$$

$$\frac{M\bar{x}}{M} = \bar{x} = \frac{\frac{158\pi}{3}}{\frac{202\pi}{5}} = \frac{395}{303} = \boxed{1.30}$$

c) $\frac{\text{mass}}{\text{volume}} = p$ so mass of $S = 2p \times$ volume
mass of cone $= p \times$ volume.
where p is a constant.

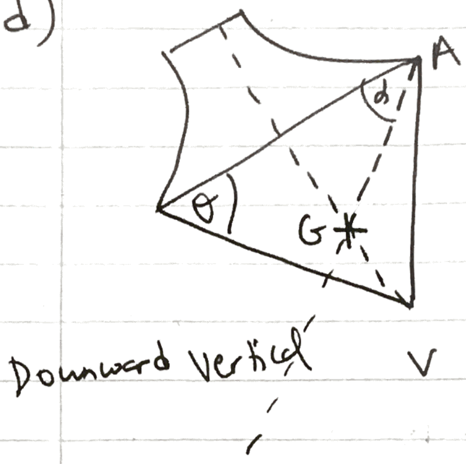
Shape	Mass (vol.)	Distance of c.o.m from V
	$2p \times \frac{202\pi}{5} = \boxed{\frac{404p\pi}{5}}$	$6 + \left(2 - \frac{395}{303}\right) = \boxed{\frac{2029}{303}}$
	$\frac{1}{3}p\pi(7)^2(6) = \boxed{98p\pi}$	$\frac{3}{4} \times 6 = \boxed{\frac{9}{2}}$
	$\boxed{\frac{894p\pi}{5}}$	$\boxed{\bar{x}}$

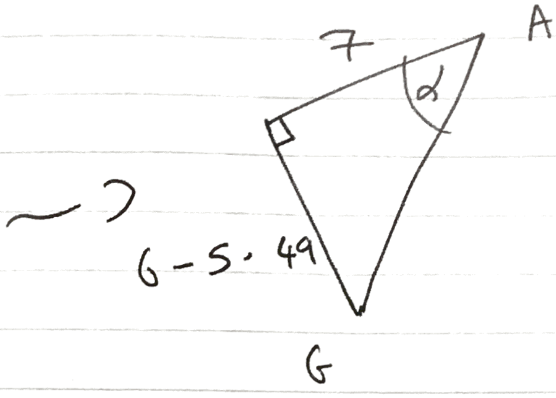
taking moments about V...

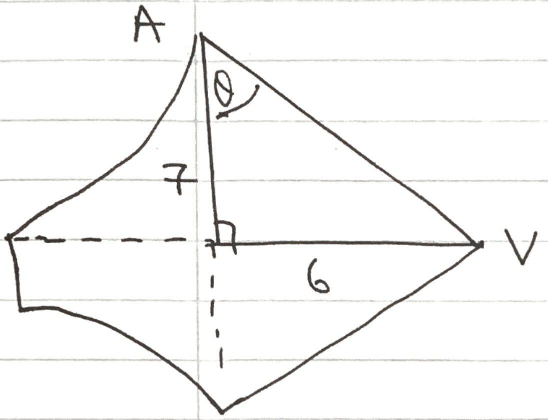
$$\frac{404}{5}\left(\frac{2029}{303}\right) + 98\left(\frac{9}{2}\right) = \frac{894}{5}(\bar{x})$$

$$\bar{x} = \frac{\frac{404}{5}\left(\frac{2029}{303}\right) + 98\left(\frac{9}{2}\right)}{\frac{894}{5}} = \boxed{5.49}$$

d)







$$\tan\alpha = \frac{6-5.49}{7}$$

$$\alpha = \tan^{-1}\left(\frac{6-5.49}{7}\right) = 4.17^\circ$$

$$\tan\theta = \frac{6}{7} \quad \therefore \theta = \tan^{-1}\left(\frac{6}{7}\right)$$

angle required $= \theta - \alpha = \tan^{-1}\left(\frac{6}{7}\right) - 4.17^\circ$

$$= \boxed{36^\circ}$$