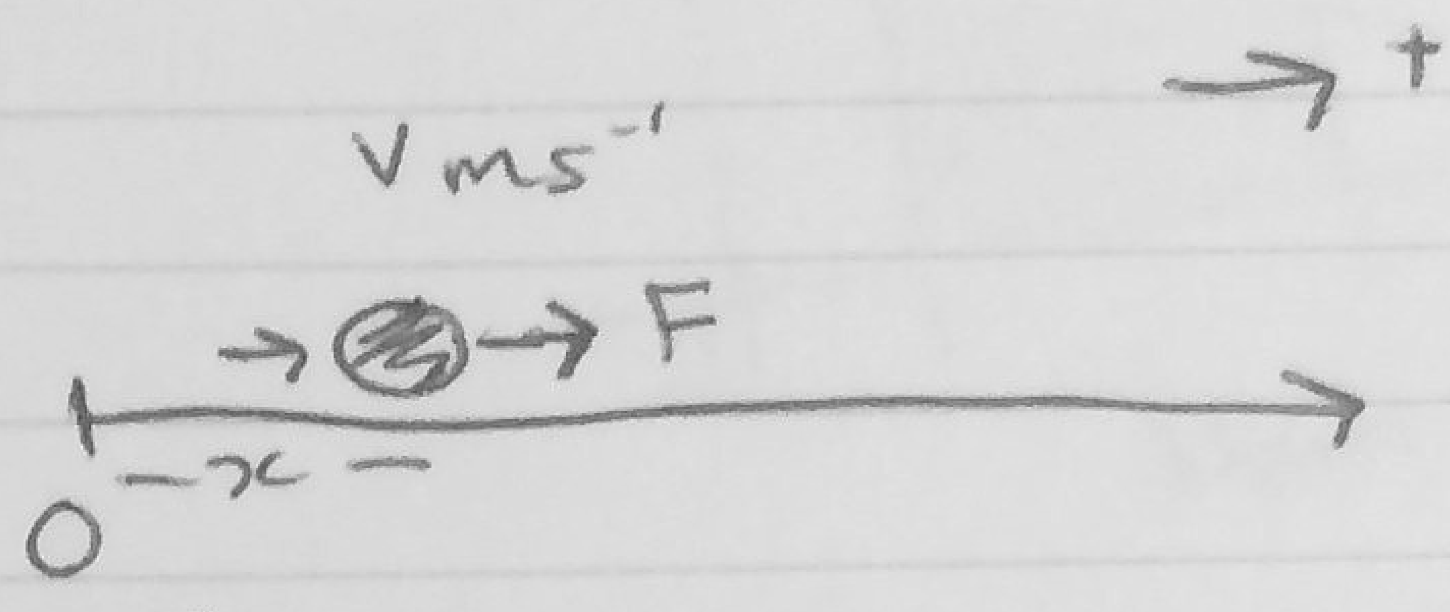
1. A particle P of mass 3 kg is moving along the horizontal x-axis. At time $t = 0$, P passes through the origin O moving in the positive x direction. At time t seconds, $OP = x$ metres and the velocity of P is v m s^{-1}. At time t seconds, the resultant force acting on P is $\frac{9}{2}(26 - x)$ N, measured in the positive x direction. For $t > 0$ the maximum speed of P is 32 m s^{-1}.

Find v^2 in terms of x.

(6)



$$a = v\frac{dv}{dx}$$

$$F = ma$$

$$\frac{9}{2}(26 - x) = 3v\frac{dv}{dx}$$

$$\int \frac{9}{2}(26 - x)\,dx = \int 3v\,dv$$

$$\frac{9}{2}\left(26x - \frac{x^2}{2}\right) = \frac{3v^2}{2} (+c)$$

max speed when $a = 0$, when $x = 26$ $v = 32$

$$\frac{3}{2}\left(26x - \frac{x^2}{2}\right) = \frac{v^2}{2} + c$$

$$\frac{3}{2}\left(26(26) - \frac{(26)^2}{2}\right) = \frac{32^2}{2} + c \qquad c = -5$$

$$\frac{v^2}{2} = \frac{3}{2}\left(26x - \frac{x^2}{2}\right) + 5$$

$$v^2 = 3\left(26x - \frac{x^2}{2}\right) + 10$$

$$v^2 = 78x - \frac{3x^2}{2} + 10$$

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2.

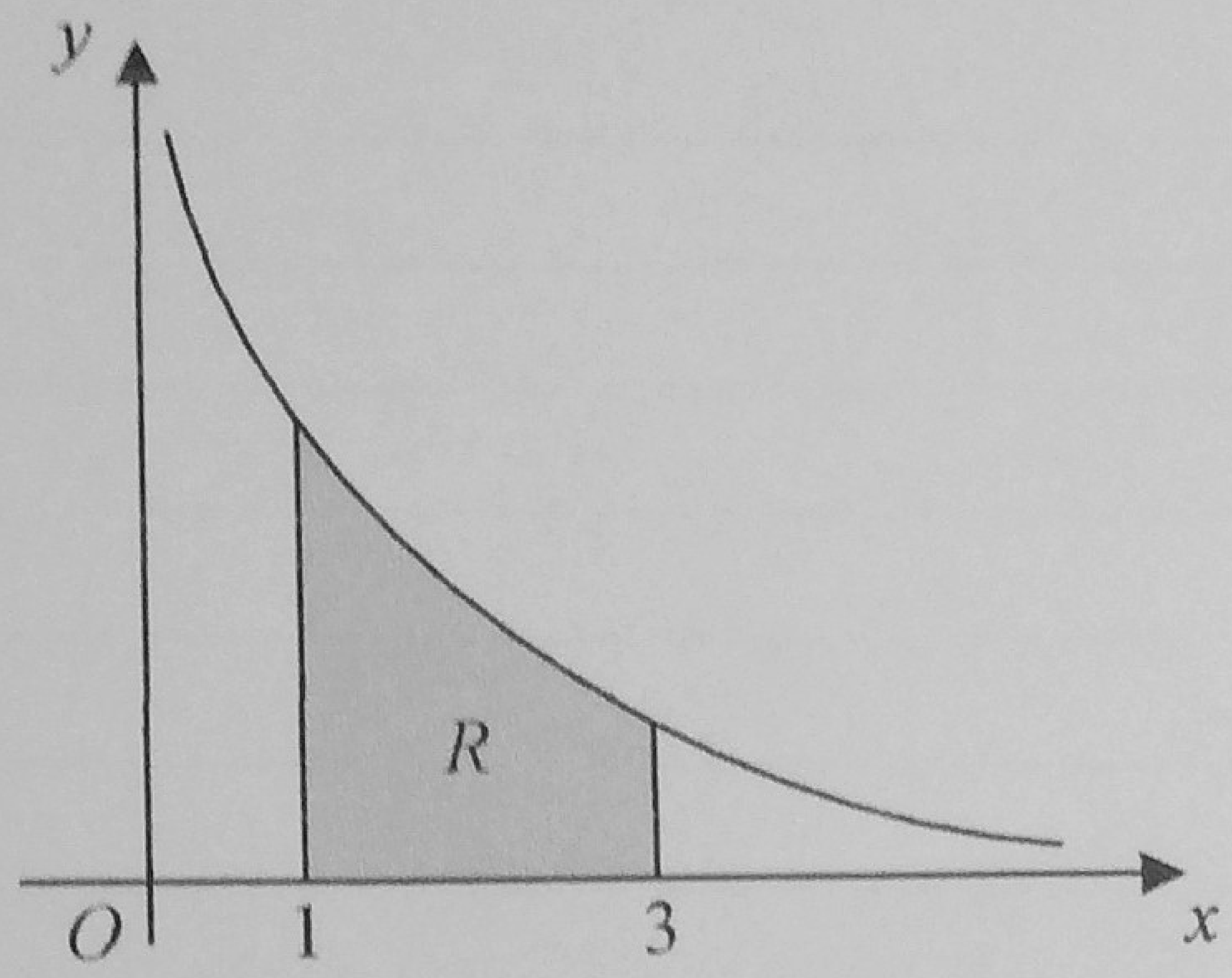

Figure 1

A uniform lamina is in the shape of the region R which is bounded by the curve with equation $y = \frac{3}{x^2}$, the lines $x = 1$ and $x = 3$, and the x-axis, as shown in Figure 1.

The centre of mass of the lamina has coordinates $(\bar{x}, \bar{y})$.

Use algebraic integration to find

(i) the value of $\bar{x}$,

(ii) the value of $\bar{y}$.

(9)

2) Area $= \int_1^3 y \, dx = \int_1^3 \frac{3}{x^2} \, dx$

$= \left[-3x^{-1}\right]_1^3 = 2$

i) $\int xy \, dx = \int_1^3 \frac{3x}{x^2} \, dx = \int_1^3 \frac{3}{x} \, dx$

$\left[3\ln x\right]_1^3 = 3\ln 3$

$\bar{x} = \frac{3\ln 3}{2}$

ii) $\int \frac{1}{2} y^2 \, dx \quad \int_1^3 \frac{1}{2} \times \frac{9}{x^4} \, dx \quad \frac{9}{2}\left[-\frac{x^{-3}}{3}\right]_1^3 = \frac{13}{9}$

$\bar{y} = \frac{13}{18}$

3.

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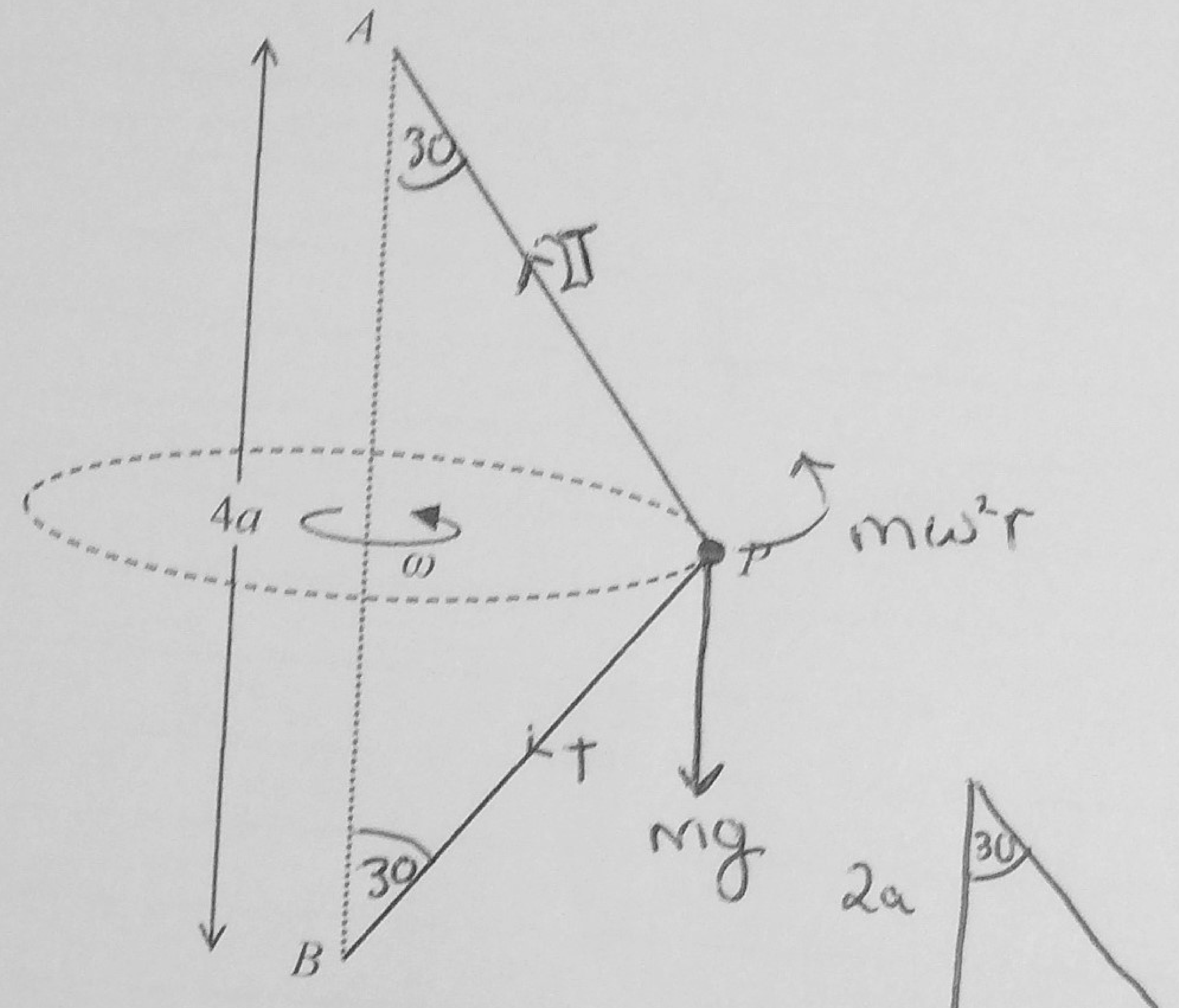


Figure 2

$\tan 30 = \dfrac{r}{2a}$

A light inextensible string has one end attached to a fixed point A and the other end attached to a particle P of mass m. An identical string has one end attached to the fixed point B, where B is vertically below A and $AB = 4a$, and the other end attached to P, as shown in Figure 2. The particle is moving in a horizontal circle with constant angular speed ω, with both strings taut and inclined at 30° to the vertical. The tension in the upper string is twice the tension in the lower string.

Find ω in terms of a and g.

(8)

R(↑) $2T\cos 30 = T\cos 30 + mg$

$\sqrt{3}T = \dfrac{\sqrt{3}}{2}T + mg$

$\dfrac{\sqrt{3}}{2}T = mg \qquad T = \dfrac{2}{\sqrt{3}}mg$

R(→) $3T\sin 30 = m\omega^2 r \qquad r = \dfrac{2}{\sqrt{3}}a$

$\dfrac{3T}{2} = m\omega^2 r$

$\dfrac{3}{\cancel{2}} \times \dfrac{\cancel{2}}{\sqrt{3}}\cancel{m}g = \cancel{m}\omega^2 \dfrac{2a}{\sqrt{3}}$

$\dfrac{3g}{\sqrt{3}} = \dfrac{2a}{\sqrt{3}}\omega^2 \qquad \dfrac{3g}{2a} = \omega^2 \qquad \omega = \sqrt{\dfrac{3g}{2a}}$

Leave blank

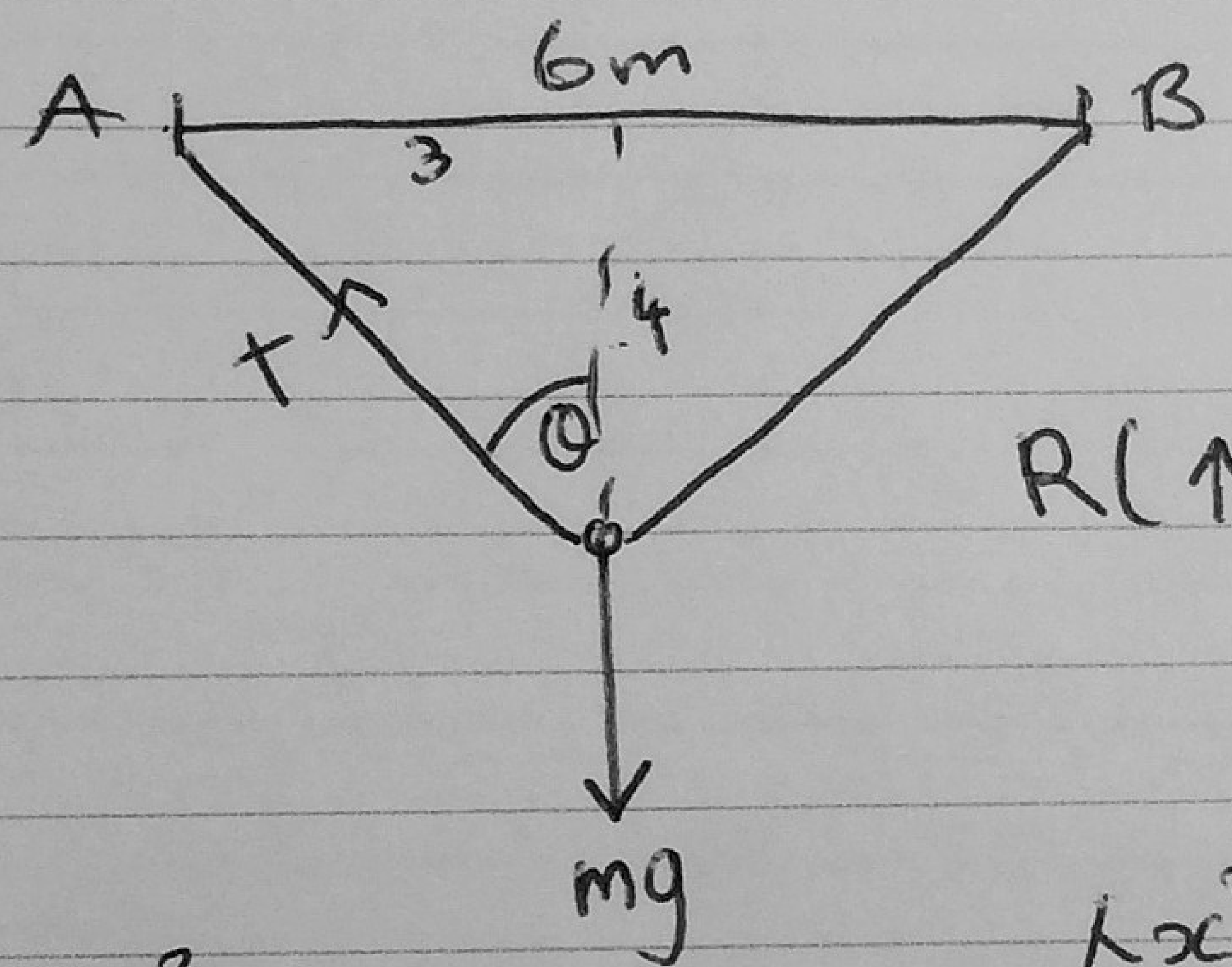
4. A light elastic string has natural length 5 m and modulus of elasticity 20 N. The ends of the string are attached to two fixed points A and B, which are 6 m apart on a horizontal ceiling. A particle P is attached to the midpoint of the string and hangs in equilibrium at a point which is 4 m below AB.

(a) Calculate the weight of P. **(6)**

The particle is now raised to the midpoint of AB and released from rest.

(b) Calculate the speed of P when it has fallen 4 m. **(5)**

4)a) $T = \frac{\lambda x}{l}$ $\quad \frac{20 \times 5}{5} = 20$



$\tan\theta = \frac{3}{4}$

R(↑) $2T\cos\alpha = mg$

$mg = 32$

$m = \frac{32}{g}$

$\frac{1}{2}mv^2 \quad mgh \quad \frac{\lambda x^2}{2l}$

b) KE = PE lost − EPE gained

$\frac{1}{2}mv^2 = 4mg - \left(\frac{20(5)^2}{2\times5} - \frac{20(1)^2}{2\times5}\right)$

$\frac{16v^2}{g} = 4g \times \frac{32}{g} - 48$

$\frac{16v^2}{g} = 128 - 48$ $\quad v^2 = 5g$

$v = 7\text{ms}^{-1}$

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5.

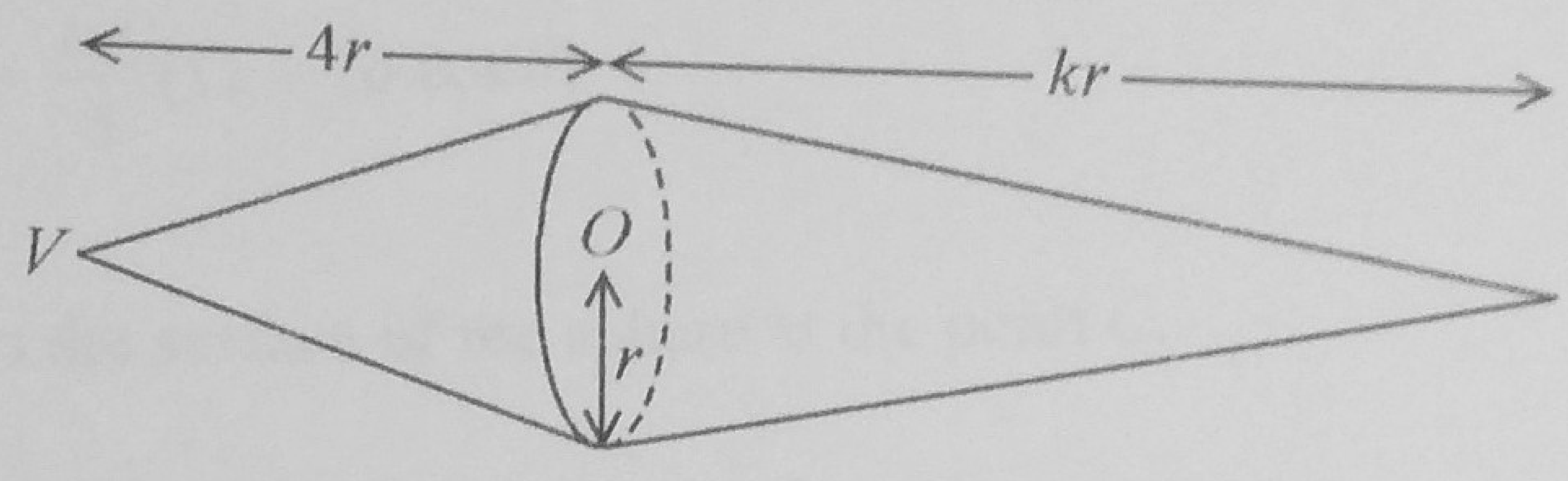


Figure 3

Figure 3 shows a uniform solid S formed by joining the plane faces of two solid right circular cones, of base radius r, so that the centres of their bases coincide at O. One cone, with vertex V, has height $4r$ and the other cone has height kr, where $k > 4$

(a) Find the distance of the centre of mass of S from O.

(4)

The point A lies on the circumference of the common base of the cones. The solid is placed on a horizontal surface with VA in contact with the surface. Given that S rests in equilibrium,

(b) find the greatest possible value of k.

(3)

When S is suspended from A and hangs freely in equilibrium, OA makes an angle of 12° with the downward vertical.

(c) Find the value of k.

(3)

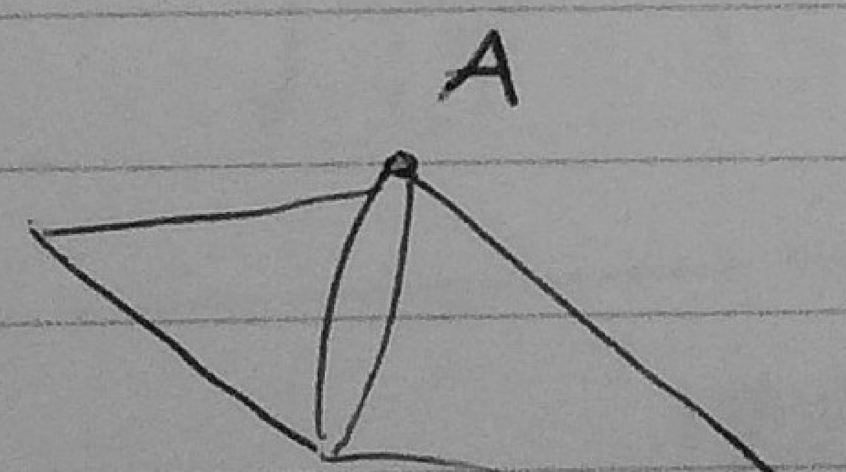
5 a)

mass	$\frac{4}{3}\pi r^3 \rho$	$\frac{1}{3}k\pi r^3\rho$	$\frac{1}{3}\pi r^3 \rho(4+k)$
Ratio	4	k	$4+k$
distance from O	$-r$	$\frac{kr}{4}$	$\bar{x}$

$$-4r + \frac{k^2 r}{4} = (4+k)\bar{x} \qquad \bar{x} = \frac{1}{4}(k-4)r$$

b) k greatest when $\frac{\bar{x}}{r} = \frac{r}{4r}$ $\quad \frac{1}{4}(k-4) = \frac{1}{4}$ $\quad k=5$

c)



$$\tan 12 = \frac{\bar{x}}{r} = \frac{1}{4}(k-4)$$

$$k = 4.8502\ldots$$

$$k = 4.9$$

Leave blank

6. A smooth sphere, with centre O and radius a, is fixed with its lowest point A on a horizontal floor. A particle P is placed on the surface of the sphere at the point B, where B is vertically above A. The particle is projected horizontally from B with speed $\sqrt{\frac{ag}{5}}$ and moves along the surface of the sphere. When OP makes an angle θ with the upward vertical, and P is still in contact with the sphere, the speed of P is v.

(a) Show that $v^2 = \frac{ag}{5}(11 - 10\cos\theta)$.

(4)

The particle leaves the surface of the sphere at the point C.

Find

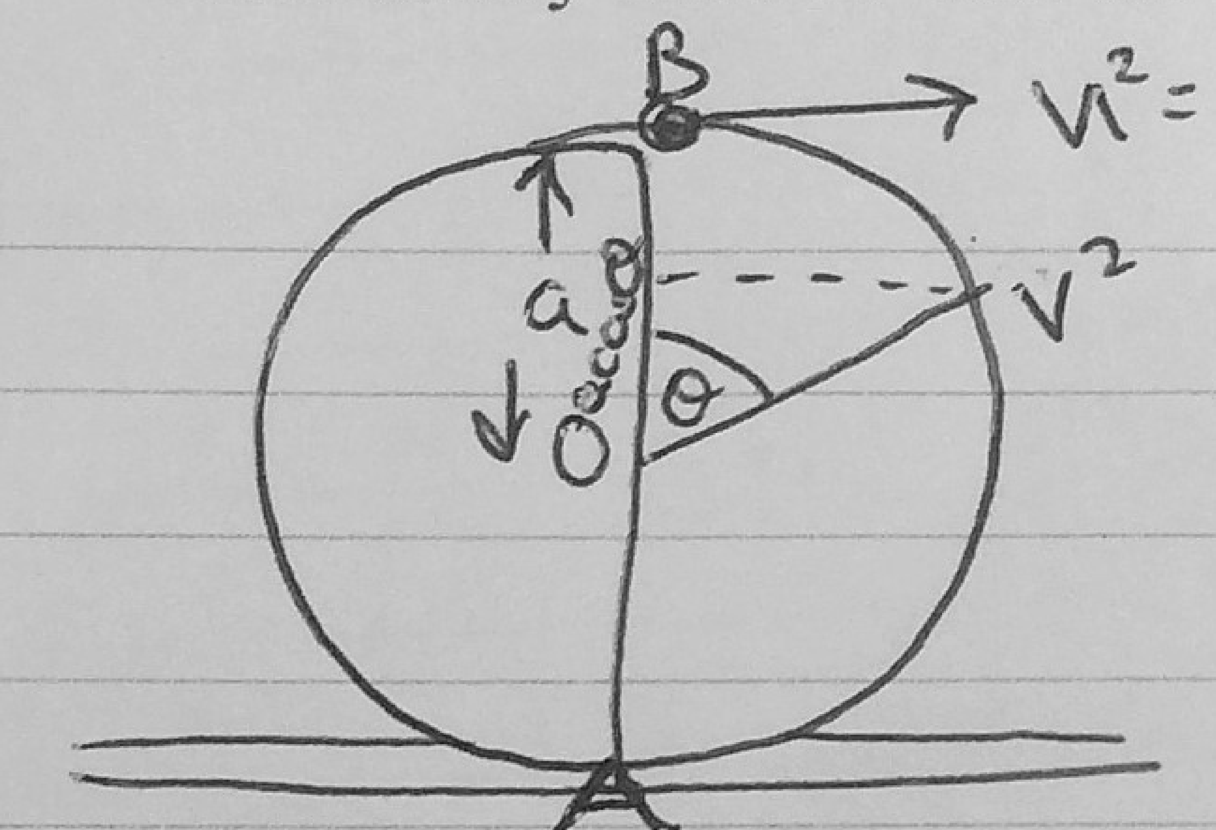
(b) the speed of P at C in terms of a and g,

(6)

(c) the size of the angle between the floor and the direction of motion of P at the instant immediately before P hits the floor.

(5)

a)



change in KE = change in PE

$\frac{1}{2}mv^2 - \frac{1}{2}mu^2 = \Delta mgh$

$\frac{1}{2}mv^2 - \frac{1}{2}m\frac{ag}{5} = mga(1-\cos\theta)$

$\frac{\cancel{m}v^2}{2} = \frac{\cancel{m}ga}{10} + \cancel{m}ga(1-\cos\theta)$

$v^2 = \frac{ga}{5} + 2ga - 2ga\cos\theta$

$v^2 = \frac{ag}{5}(11 - 10\cos\theta)$

b) $\cancel{m}g\cos\alpha = \frac{\cancel{m}v^2}{a}$ $\quad g\cos\alpha = \frac{g}{5}(11 - 10\cos\alpha)$

$\cos\alpha = \frac{11}{15}$

$v^2 = \frac{ag}{5}\left(11 - 10\left(\frac{11}{15}\right)\right) = \sqrt{\frac{11ag}{15}}$

c) $\rightarrow \sqrt{\frac{11ag}{15}}\cos\alpha \quad \sqrt{\frac{11ag}{15}} \times \frac{11}{15}$

energy conservation $\quad 2mga = \frac{1}{2}mV^2 - \frac{1}{2}m\frac{ag}{5} \quad V^2 = \frac{21ag}{5}$

$\cos\theta = \sqrt{\frac{11ag}{15}} \times \frac{11}{15} \times \sqrt{\frac{5}{21ag}}$ $\quad \underline{\underline{\theta = 72^\circ}}$

7. A particle P of mass m is attached to one end of a light elastic string, of natural length a and modulus of elasticity λ. The other end of the string is attached to a fixed point A on a smooth plane which is inclined at 30° to the horizontal. The string lies along a line of greatest slope of the plane. The particle rests in equilibrium at the point B, where B is lower than A and $AB = \frac{6}{5}a$.

(a) Show that $\lambda = \frac{5}{2}mg$.

(4)

The particle is now pulled down a line of greatest slope to the point C, where $BC = \frac{1}{5}a$, and released from rest.

(b) Show that P moves with simple harmonic motion of period $2\pi\sqrt{\frac{2a}{5g}}$

(6)

(c) Find, in terms of g, the greatest magnitude of the acceleration of P while the string is taut.

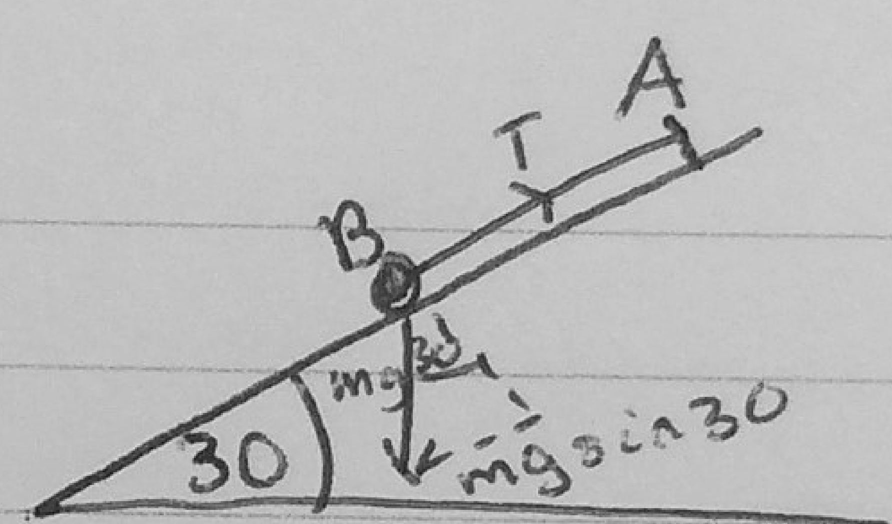
(2)

The midpoint of BC is D and the string becomes slack for the first time at the point E.

(d) Find, in terms of a and g, the time taken by P to travel directly from D to E.

(4)

a) $T = \frac{\lambda \times a/5}{a}$



$\frac{mg}{2} = \frac{\lambda}{5}$ $\quad \lambda = \frac{5mg}{2}$ $\quad T = mg\sin 30 = \frac{mg}{2}$

b) length $\frac{6a}{5} + x$

$\frac{1}{2}mg - \frac{5}{2}mg\left(\frac{a/5 + x}{a}\right) = m\ddot{x}$ $\quad T = \frac{2\pi}{\omega}$

$\omega^2 = \frac{5g}{2a}$

$\ddot{x} = -\frac{5g}{2a}x$ ∴ SHM

$T = 2\pi\sqrt{\frac{2a}{5g}}$

c) $a_{max} = \omega^2 A$

$\frac{5g}{2\cancel{a}} \times \frac{\cancel{a}}{\cancel{5}} = \frac{g}{2}$

d) $x = \frac{a}{5}\sin\omega t$ $\quad \omega t = \frac{\pi}{6}$ $\quad t = \frac{\pi}{6\omega} = \frac{\pi}{6}\sqrt{\frac{2a}{5g}}$

$\frac{a}{10} = \frac{a}{5}\sin\omega t$ $\quad$ total time

$\frac{\pi}{6}\sqrt{\frac{2a}{5g}} + \frac{\pi}{2}\sqrt{\frac{2a}{5g}} = \frac{2\pi}{3}\sqrt{\frac{2a}{5g}}$ s