M3 January 2014 IAL (MA)

Q1) $v = \sqrt{8x^{\frac{3}{2}} - 4} = (8x^{\frac{3}{2}} - 4)^{\frac{1}{2}}$

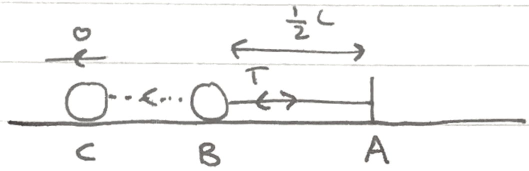
$\frac{dv}{dx} = \frac{1}{2}(8x^{\frac{3}{2}} - 4)^{-\frac{1}{2}}(12x^{\frac{1}{2}}) = \frac{6\sqrt{x}}{\sqrt{8x^{3/2} - 4}}$ //

$a = v\frac{dv}{dx} = \cancel{\sqrt{8x^{3/2} - 4}} \times \frac{6\sqrt{x}}{\cancel{\sqrt{8x^{3/2} - 4}}} = 6\sqrt{x}$ //

$\sum F = ma$: $F = 3\sqrt{x}$.

at $x = 4$, $F = 3\sqrt{4} = \boxed{6\text{ N}}$

Q2)



At B : $KE = 0$

$EPE = \frac{\cancel{2}mg(\frac{l}{2})^2}{\cancel{2}l} = \frac{mgl^{\cancel{2}}}{4\cancel{l}} = \frac{mgl}{4}$ //

At C : $KE = 0$

$EPE = \frac{\cancel{2}mg}{\cancel{2}l}(d - \frac{l}{2})^2$

+ W.D due to friction $= \frac{1}{4}(mg)d$.

So by C.o.E : $\frac{\cancel{m}gl}{4} = \frac{\cancel{m}g}{l}(d - \frac{l}{2})^2 + \frac{\cancel{m}gd}{4}$

$$\frac{l}{4} = \frac{1}{l}\left(d - \frac{l}{2}\right)^2 + \frac{d}{4}$$

$$\frac{l}{4} = \frac{1}{l}\left(d^2 - dl + \frac{l^2}{4}\right) + \frac{d}{4}$$

$$\underline{\times l}: \quad \cancel{\frac{l^2}{4}} = d^2 - dl + \cancel{\frac{l^2}{4}} + \frac{dl}{4}$$

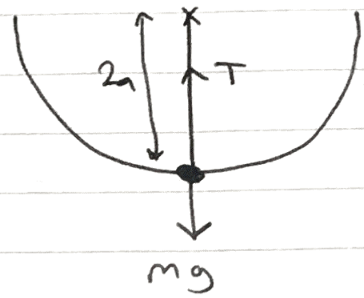
$$\Rightarrow \quad d^2 - ld + \frac{dl}{4} = 0$$

$$\Rightarrow \quad d\left(d - l + \frac{l}{4}\right) = 0$$

$$\Rightarrow \quad d - l + \frac{l}{4} = 0$$

$$\Rightarrow \quad \boxed{d = \frac{3l}{4}} \quad = \text{distance from } B.$$

Q3) Greatest tension is at the lowest point.

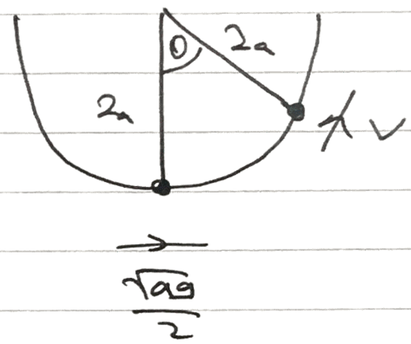


N2L $\uparrow^{+}$ (P): $T - mg = \frac{mv^2}{r}$

$$\frac{9mg}{8} - mg = \frac{mv^2}{2a}$$

$$\frac{g}{8}(2a) = v^2 = \frac{ag}{4} \;//$$

using C.O.E from lowest point to when P makes an angle θ with downward vertical,



Initially: $KE = \frac{1}{2}m\left(\frac{ag}{4}\right) = \frac{amg}{8}$

$GPE = 0$

Finally: $KE = \frac{1}{2}m\left(\frac{ag}{20}\right)$

$GPE = mg(2a - 2a\cos\theta)$

$$\Rightarrow \frac{amg}{8} = \frac{amg}{40} + 2amg(1-\cos\theta)$$

$$\Rightarrow \frac{\frac{1}{8} - \frac{1}{40}}{2} = 1 - \cos\theta = \frac{1}{20}$$

$$\therefore \cos\theta = \frac{19}{20} \;//$$

$$\theta = \cos^{-1}\frac{19}{20} = \boxed{18°}$$

Q4a) $V = \pi \int_0^1 y^2\,dx = \pi \int_0^1 [e^{-2x}]\,dx$

$= \left[-\frac{1}{2}\pi e^{-2x}\right]_0^1 = \left[-\frac{1}{2}\pi e^{-2}\right] + \left[\frac{1}{2}\pi\right]$

$= \frac{\pi}{2}(1 - e^{-2})$

b) $M\bar{x} = \pi \int_0^1 y^2 x\,dx = \pi \int_0^1 [xe^{-2x}]\,dx$

By Parts : $\frac{dv}{dx} = e^{-2x} \rightarrow v = -\frac{1}{2}e^{-2x}$

$u = x \rightarrow u' = 1$

$\Rightarrow \left[-\frac{1}{2}xe^{-2x}\right]_0^1 + \frac{1}{2}\int_0^1 (e^{-2x})\,dx$

$\Rightarrow \left[-\frac{1}{2}e^{-2}\right] - \left[\frac{1}{4}e^{-2x}\right]_0^1$

$\Rightarrow -\frac{1}{2}e^{-2} - \left[\frac{1}{4}e^{-2} - \frac{1}{4}\right]$

$\Rightarrow -\frac{3}{4}e^{-2} + \frac{1}{4}$

$\therefore M\bar{x} = \pi\left[\frac{1}{4} - \frac{3}{4}e^{-2}\right]$ //

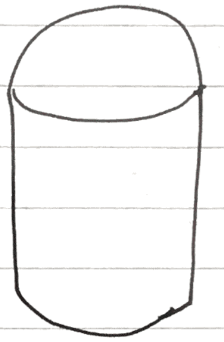
and from (a), $M = \frac{\pi}{2}(1-e^{-2})$ //

so $\bar{x} = \dfrac{\cancel{\pi}\left[\frac{1}{4} - \frac{3}{4}e^{-2}\right]}{\frac{\cancel{\pi}}{2}(1-e^{-2})} = \dfrac{\frac{1}{2} - \frac{3}{2}e^{-2}}{1-e^{-2}}$

$= \dfrac{1-3e^{-2}}{2-2e^{-2}} \overset{(\times e^2)}{\underset{(\times e^2)}{=}} \dfrac{e^2-3}{2e^2-2}$

$= \dfrac{e^2-3}{2(e^2-1)}$

Q5a)

Shape	Mass (vol.)	Distance of c.o.m from O
	$\frac{2}{3}\pi r^3(3u)$ $= \boxed{2u\pi r^3}$	$\frac{3r}{8} + 3r = \boxed{\frac{27}{8}r}$
	$\pi r^2(3r)(u)$ $= \boxed{3\pi r^3 u}$	$\boxed{\frac{3r}{2}}$
	$\boxed{5\pi r^3 u}$	$\boxed{\bar{y}}$

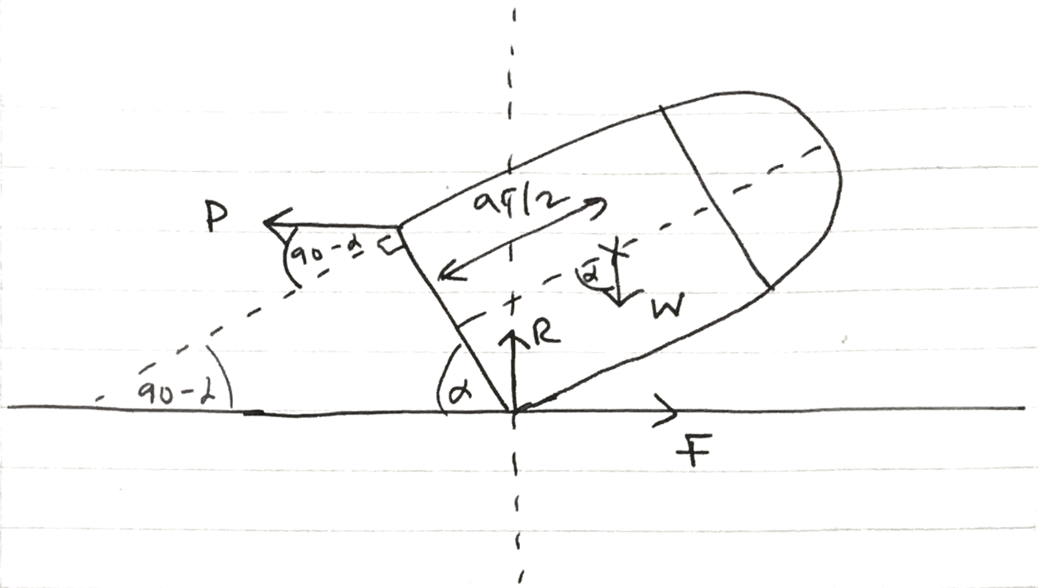
taking moments about diameter through O,

$$\cancel{2\pi r^3 u}\left(\frac{27r}{8}\right) + \cancel{3\pi r^3 u}\left(\frac{3r}{2}\right) = \cancel{5\pi r^3 u}(\bar{y})$$

$$\frac{27r}{4} + \frac{9r}{2} = 5\bar{y} = \frac{45r}{4}$$

$$\therefore \bar{y} = \frac{45r}{5 \times 4} = \boxed{\frac{9r}{4}}$$

5b)



$$\left.\begin{array}{l} R(\updownarrow):\ W = R \\ \\ R(\leftrightarrow):\ F = P \end{array}\right\} \quad \begin{array}{l} F = \mu R \\ P = \mu W. \end{array}$$

$$M(\text{Point of contact}):\quad W\left[\sin\alpha\left(\frac{9r}{4}\right) - r\cos\alpha\right] = P(\cos(90-\alpha)) \times 2r$$

$$\Rightarrow \quad W\cancel{r}\left(\frac{9r}{4}\sin\alpha - \cos\alpha\right) = 2P\cancel{r}\sin\alpha$$

$$\underline{\div \sin\alpha}: \quad W\left(\frac{9r}{4} - \cot\alpha\right) = 2P$$

$$\therefore \quad P = \frac{W}{2}\left(\frac{9r}{4} - \cot\alpha\right)$$

and $\quad P = \mu W$

$$\Rightarrow \quad \frac{\cancel{W}}{2}\left(\frac{9r}{4} - \cot\alpha\right) = \mu \cancel{W}$$

$$\Rightarrow \quad \mu = \frac{9r}{8} - \frac{1}{2}\cot\alpha = \frac{1}{8}(9r - 4\cot\alpha)$$

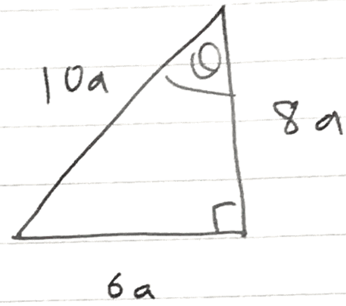
<u>REMEMBER</u>: We are taking moments about the dotted line through the point of contact
So find perpendicular distances from the lines of action

of P/F to the dotted line.

Q6a) $\sqrt{(10a)^2-(8a)^2} = 6a$ //

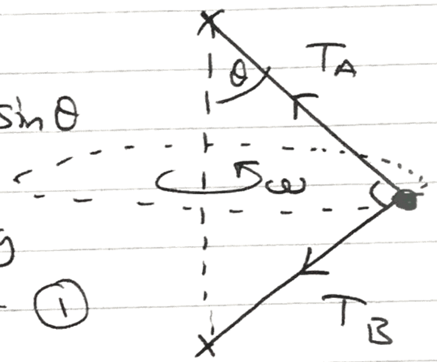
hence $APB = 90°$

(Pythagoras).



b) $R(\updownarrow)$: $T_A\cos\theta = mg + T_B\sin\theta$

$\therefore T_A\cos\theta - T_B\sin\theta = mg$ — ①



$\cos\theta = \frac{8}{10}$

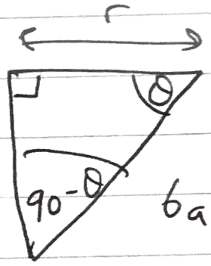
$\sin\theta = \frac{6}{10}$

$\tan\theta = \frac{3}{4}$

N2L (P) : $T_A\sin\theta + T_B\cos\theta = mr\omega^2$

$r = 6a\cos\theta$

$r = 6a \times \frac{4}{5} = \frac{24a}{5}$ //



$\therefore T_A\sin\theta + T_B\cos\theta = \frac{24ma\omega^2}{5}$

②

now remember that the lower part of the string must remain taut for P to continue moving in circular motion.

so $T_B \geq 0$.

solve ① and ② for T_B:

$\underline{① \div \cos\theta}$: $T_A - T_B \tan\theta = \frac{5mg}{4}$

$$T_A - \frac{3}{4}T_B = \frac{5mg}{4}$$

$\underline{② \div \cos\theta}$: $T_A\left(\frac{3}{4}\right) + T_B = 6ma\omega^2$

$\times 4/3$

$\Rightarrow \quad T_A + \frac{4}{3}T_B = 8ma\omega^2$

$\Rightarrow \quad T_A = 8ma\omega^2 - \frac{4}{3}T_B$

$\underline{\text{from } ①}$: $T_A = \frac{3}{4}T_B + \frac{5mg}{4}$

$\underline{① = ②}$: $\frac{3T_B}{4} = -\frac{4}{3}T_B - \frac{5mg}{4} + 8ma\omega^2$

$$\frac{25}{12}T_B = 8ma\omega^2 - \frac{5mg}{4}$$

$$\frac{25}{12}T_B = \frac{m}{4}\left(32a\omega^2 - 5g\right)$$

$$\therefore T_B = \frac{3m}{25}\left(32a\omega^2 - 5g\right)$$

$T_B \geq 0$

$32a\omega^2 - 5g \geq 0$

$\omega^2 \geq \frac{5g}{32a} \quad \therefore \omega \geq \sqrt{\frac{5g}{32a}}$

and $T = \frac{2\pi}{\omega} \longrightarrow \omega = \frac{2\pi}{T}$

$$\therefore \sqrt{\frac{5g}{32a}} \leq \frac{2\pi}{T}$$

take reciprocal of both sides so signs flip...

$$T \leq \frac{2\pi}{\sqrt{\frac{5g}{32a}}} \rightarrow T \leq 2\pi\sqrt{\frac{32a}{5g}}$$

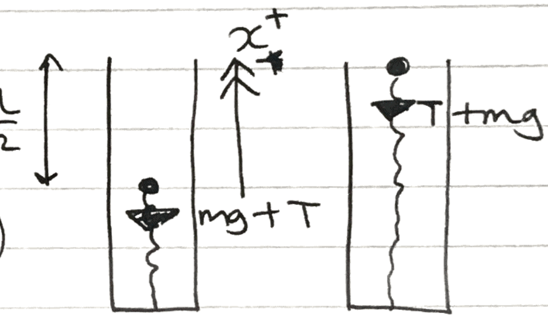
(max time is $2\pi\sqrt{\frac{32a}{5g}}$)

Q7a) $T = mg$

$$\frac{8\cancel{m}\cancel{g}e}{\cancel{L}L} = \cancel{m}\cancel{g}$$

$$\therefore 8e = L \qquad \therefore \boxed{\frac{L}{8} = e}$$

b) N2L(P) $\uparrow^{+}$: $-mg - T = m\ddot{x}$



$$T = \frac{\lambda x}{L} = \frac{8mg}{L}\left(x - \frac{L}{8}\right)$$

where x is distance from equilibrium position,

$$\therefore -mg - \frac{8mg}{L}\left(x - \frac{L}{8}\right) = m\ddot{x}$$

$$-mg - \frac{8mgx}{L} + mg = m\ddot{x}$$

$$\therefore -\frac{8gx}{L} = \ddot{x} \quad //$$

so $\ddot{x} = -\left(\frac{8g}{l}\right)x$ hence P moves with S.H.M.

$$\omega^2 = \frac{8g}{l} \longrightarrow \omega = \sqrt{\frac{8g}{l}}$$

$$T = 2\pi \times \frac{1}{\omega} = 2\pi \times \sqrt{\frac{l}{8g}}$$

c) Initially : $KE = 0$

$GPE = 0$

$EPE = \frac{8mg}{2l}\left(\frac{l}{2}\right)^2$

Finally : $KE = \frac{1}{2}mv^2$

$GPE = mg\left(\frac{l}{2}\right)$

$EPE = \frac{8mg}{2l}(0) = 0$ //

$$\Rightarrow \frac{8\cancel{m}g}{2l}\left(\frac{l^2}{4}\right) = \frac{\cancel{m}v^2}{2} + \frac{\cancel{m}gl}{2}$$

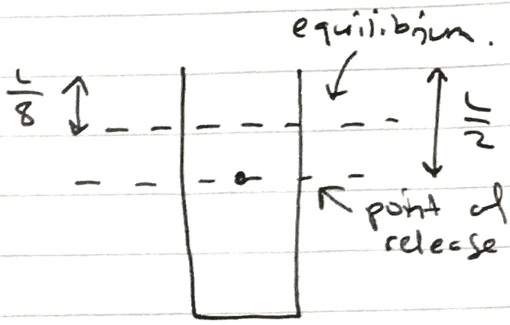
$$\Rightarrow \frac{\cancel{4}\,gl^{\cancel{2}}}{\cancel{4}\cancel{l}} = \frac{v^2}{2} + \frac{gl}{2}$$

$$\Rightarrow gl - \frac{gl}{2} = \frac{v^2}{2} = \frac{gl}{2}$$

$$\therefore v^2 = gl \rightarrow \boxed{v = \sqrt{gl} = u}$$

d) the fact that P is released at an 'endpoint' means the characteristic S.H.M equation to use is: $x = a\cos\omega t$.

$\dot{x} = -a\omega\sin\omega t$.



$$a = \frac{l}{2} - \frac{l}{8} = \frac{3l}{8}$$

$$\therefore \dot{x} = -\frac{3l}{8}\sqrt{\frac{8g}{l}}\sin\left(t\sqrt{\frac{8g}{l}}\right).$$

$$-\sqrt{\frac{9\cancel{g}\cancel{l}}{32}} = \frac{-3}{8}\sqrt{8\cancel{g}\cancel{l}}\sin\left(t\sqrt{\frac{8g}{l}}\right)$$

$$\frac{-\sqrt{\frac{9}{32}}}{-\frac{3}{8}\sqrt{8}} = \sin\left(t\sqrt{\frac{8g}{l}}\right) = \frac{1}{2}$$

x will be negative as P will below the equilibrium point when it first attains a speed of $\sqrt{\frac{9gl}{32}}$.

$$\therefore t\sqrt{\frac{8g}{l}} = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$$

$$\therefore \frac{\pi}{6} \times \sqrt{\frac{l}{8g}} = t$$