Pearson Edexcel

Mark Scheme (Results)

Summer 2021

Pearson Edexcel International A Level
In Further Pure Mathmatics F3
(WFM03/01)

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General Marking Guidance

- All candidates must receive the same treatment. Examiners must mark the first candidate in exactly the same way as they mark the last.
- Mark schemes should be applied positively. Candidates must be rewarded for what they have shown they can do rather than penalised for omissions.
- Examiners should mark according to the mark scheme not according to their perception of where the grade boundaries may lie.
- There is no ceiling on achievement. All marks on the mark scheme should be used appropriately.
- All the marks on the mark scheme are designed to be awarded. Examiners should always award full marks if deserved, i.e. if the answer matches the mark scheme. Examiners should also be prepared to award zero marks if the candidate's response is not worthy of credit according to the mark scheme.
- Where some judgement is required, mark schemes will provide the principles by which marks will be awarded and exemplification may be limited.
- When examiners are in doubt regarding the application of the mark scheme to a candidate's response, the team leader must be consulted.
- Crossed out work should be marked UNLESS the candidate has replaced it with an alternative response.

EDEXCEL IAL MATHEMATICS

General Instructions for Marking

1. The total number of marks for the paper is 75.

2. The Edexcel Mathematics mark schemes use the following types of marks:

- **M** marks: method marks are awarded for 'knowing a method and attempting to apply it', unless otherwise indicated.
- **A** marks: Accuracy marks can only be awarded if the relevant method (M) marks have been earned.
- **B** marks are unconditional accuracy marks (independent of M marks)
- Marks should not be subdivided.

3. Abbreviations

These are some of the traditional marking abbreviations that will appear in the mark schemes.

- bod – benefit of doubt
- ft – follow through
- the symbol $\sqrt{}$ will be used for correct ft
- cao – correct answer only
- cso - correct solution only. There must be no errors in this part of the question to obtain this mark
- isw – ignore subsequent working
- awrt – answers which round to
- SC: special case
- oe – or equivalent (and appropriate)
- dep – dependent
- indep – independent
- dp decimal places
- sf significant figures
- * The answer is printed on the paper
- ☐ The second mark is dependent on gaining the first mark

4. All A marks are 'correct answer only' (cao.), unless shown, for example, as A1 ft to indicate that previous wrong working is to be followed through. After a misread however, the subsequent A marks affected are treated as A ft, but manifestly absurd answers should never be awarded A marks.

5. For misreading which does not alter the character of a question or materially simplify it, deduct two from any A or B marks gained, in that part of the question affected.

6. If a candidate makes more than one attempt at any question:
- If all but one attempt is crossed out, mark the attempt which is NOT crossed out.
- If either all attempts are crossed out or none are crossed out, mark all the attempts and score the highest single attempt.

7. Ignore wrong working or incorrect statements following a correct answer.

General Principles for Further Pure Mathematics Marking

(But note that specific mark schemes may sometimes override these general principles).

Method mark for solving 3 term quadratic:

1. Factorisation

$(x^2 + bx + c) = (x + p)(x + q)$, where $|pq| = |c|$, leading to x = ...

$(ax^2 + bx + c) = (mx + p)(nx + q)$, where $|pq| = |c|$ and $|mn| = |a|$, leading to x = ...

2. Formula

Attempt to use the correct formula (with values for a, b and c).

3. Completing the square

Solving $x^2 + bx + c = 0$: $\left(x \pm \frac{b}{2}\right)^2 \pm q \pm c = 0, \quad q \neq 0$, leading to x = ...

Method marks for differentiation and integration:

1. Differentiation

Power of at least one term decreased by 1. ($x^n \rightarrow x^{n-1}$)

2. Integration

Power of at least one term increased by 1. ($x^n \rightarrow x^{n+1}$)

Use of a formula

Where a method involves using a formula that has been learnt, the advice given in recent examiners' reports is that the formula should be quoted first.

Normal marking procedure is as follows:

Method mark for quoting a correct formula and attempting to use it, even if there are small errors in the substitution of values.

Where the formula is not quoted, the method mark can be gained by implication from correct working with values, but may be lost if there is any mistake in the working.

Exact answers

Examiners' reports have emphasised that where, for example, an exact answer is asked for, or working with surds is clearly required, marks will normally be lost if the candidate resorts to using rounded decimals.

Answers without working

The rubric says that these may not gain full credit. Individual mark schemes will give details of what happens in particular cases. General policy is that if it could be done "in your head", detailed working would not be required. Most candidates do show working, but there are occasional awkward cases and if the mark scheme does not cover this, please contact your team leader for advice.

Summary of changes from Provisional Mark Scheme

A few minor changes were made to the Mark Scheme before marking on the marking service began.

Question Number	**Summary of changes**
Q03b	The B1 for a correct matrix of minors was changed to an M1 to be fairer to the candidates so that a small slip in the matrix of minors was not penalised as heavily.
Q04ii	The method marks were made dependent on each other as candidates were unlikely to make significant progress unless the first derivative was of the correct form and some first derivatives were trivial expressions.
Q06c	Alternative 3 added
Q07ii	Alternative added

Question Number	Scheme	Notes	Marks
1(a)	$1-\tanh^2 x \equiv \text{sech}^2 x$		
	$1-\tanh^2 x = 1-\left(\frac{e^x-e^{-x}}{e^x+e^{-x}}\right)^2$	Replaces the tanh x on the lhs with a **correct** expression in terms of exponentials.	B1
	$=\frac{(e^x+e^{-x})^2-(e^x-e^{-x})^2}{(e^x+e^{-x})^2}=\frac{(e^{2x}+2+e^{-2x})-(e^{2x}-2+e^{-2x})}{(e^x+e^{-x})^2}$ or e.g. $\frac{2e^{2x}\times 2e^{-2x}}{(e^x+e^{-x})^2}$ Attempts to find common denominator and expand numerator		M1
	$=\left(\frac{4}{(e^x+e^{-x})^2}\right)=\text{sech}^2 x$ *	Obtains the rhs with no errors.	A1cso
			(3)
ALT 1	$1-\tanh^2 x = (1-\tanh x)(1+\tanh x)$ $=\left(1-\left(\frac{e^x-e^{-x}}{e^x+e^{-x}}\right)\right)\left(1+\left(\frac{e^x-e^{-x}}{e^x+e^{-x}}\right)\right)$	Uses the difference of 2 squares on the lhs and replaces the tanh x with a **correct** expression in terms of exponentials.	B1
	$=\left(\frac{2e^{-x}}{e^x+e^{-x}}\right)\left(\frac{2e^x}{e^x+e^{-x}}\right)$	Attempt to find common denominators and simplify numerators.	M1
	$=\left(\frac{4}{(e^x+e^{-x})^2}\right)=\text{sech}^2 x$ *	Obtains the rhs with no errors.	A1cso
ALT 2	$\text{sech}^2 x = \frac{4}{(e^x+e^{-x})^2}$	Replaces the sech x on the rhs with a **correct** expression in terms of exponentials.	B1
	$=\frac{(e^{2x}+2+e^{-2x})-(e^{2x}-2+e^{-2x})}{(e^x+e^{-x})^2}=\frac{(e^x+e^{-x})^2-(e^x-e^{-x})^2}{(e^x+e^{-x})^2}$ Attempts to express the "4" in terms of the denominator.		M1
	$=1-\left(\frac{e^x-e^{-x}}{e^x+e^{-x}}\right)^2=1-\tanh^2 x$ *	Obtains the lhs with no errors.	A1cso

(b)	$2\operatorname{sech}^2 x + 3\tanh x = 3 \Rightarrow 2(1-\tanh^2 x) + 3\tanh x = 3$ $\Rightarrow 2\tanh^2 x - 3\tanh x + 1 = 0$ Uses $\operatorname{sech}^2 x = 1 - \tanh^2 x$ and forms a 3 term quadratic in $\tanh x$		M1
	$(2\tanh x - 1)(\tanh x - 1) = 0 \Rightarrow \tanh x = \ldots$	Solves 3TQ by any valid method including calculator.	M1
	$\tanh x = \frac{1}{2} \to x = \ln\sqrt{3}$	$\ln\sqrt{3}$. Accept $\frac{1}{2}\ln 3,\ -\frac{1}{2}\ln\frac{1}{3}$ And no other answers.	A1
			(3)
ALT	$2\operatorname{sech}^2 x + 3\tanh x = 3 \Rightarrow 2\left(\frac{4}{(e^x + e^{-x})^2}\right) + 3\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) = 3$ $\Rightarrow 8 + 3(e^{2x} - e^{-2x}) = 3(e^{2x} + 2 + e^{-2x}) \Rightarrow \ldots$ Substitutes the correct exponential forms, attempts to eliminate fractions and collect terms		M1
	$6e^{-2x} = 2 \Rightarrow e^{-2x} = \frac{1}{3}$	Rearranges to reach $e^{-2x} = \ldots$	M1
	$x = \ln\sqrt{3}$	$\ln\sqrt{3}$. Accept $\frac{1}{2}\ln 3,\ -\frac{1}{2}\ln\frac{1}{3}$ And no other answers.	A1
			Total 6

Question Number	Scheme	Notes	Marks
2.	$y=\sqrt{9-x^2}$, $0\le x\le 3$		
(a)	$\frac{dy}{dx}=-\frac{x}{\sqrt{9-x^2}}$	Correct derivative in any form.	B1
	Note that the derivative may be obtained implicitly after squaring e.g. $y=\sqrt{9-x^2}\Rightarrow y^2=9-x^2\Rightarrow 2y\frac{dy}{dx}=-2x\Rightarrow\frac{dy}{dx}=-\frac{x}{\sqrt{9-x^2}}$		
	Length of $C=\int\sqrt{1+\frac{x^2}{9-x^2}}\,dx$	Uses $\int\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ with their $\frac{dy}{dx}$	M1
	Note that the above may be obtained via the implicit route as e.g. $\int\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int\sqrt{1+\frac{x^2}{y^2}}\,dx=\int\sqrt{1+\frac{x^2}{9-x^2}}\,dx$ In which case the B1 is implied.		
	$=\int\sqrt{\frac{9}{9-x^2}}\,dx=3\arcsin\frac{x}{3}(+c)$ $\left(\text{or } -3\arccos\frac{x}{3}(+c)\right)$ $\int_0^3\sqrt{\frac{9}{9-x^2}}\,dx=3\arcsin(1)-3\arcsin(0)$ $\left(\text{or } -3\arccos(1)+3\arccos(0)\right)$ Finds common denominator, integrates to obtain arcsin... or arccos... and applies the limits 0 and 3.		M1
	$=\frac{3\pi}{2}$ *	Obtains the printed answer with no errors. This mark should be withheld if there is no evidence at all of the limits being applied.	A1
	Special case: If $+\frac{x}{\sqrt{9-x^2}}$ is obtained for $\frac{dy}{dx}$ score B0M1M1A1 if otherwise correct but allow full recovery in (b)		
			(4)
(b)	Surface Area $=\int 2\pi\sqrt{9-x^2}\left(\sqrt{\frac{9}{9-x^2}}\right)dx$	Uses $\int 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ with their $\frac{dy}{dx}$	M1
	$=\int_0^3 6\pi\,dx=6\pi[x]_0^3=\ldots$	Integrates to obtain kx and applies the limits 0 and 3. Condone omission of the lower limit.	M1
	$=18\pi$	18π cao	A1
			(3)
			Total 7

Question Number	Scheme	Notes	Marks
3.	$\mathbf{M} = \begin{pmatrix} 3 & 1 & p \\ 1 & 1 & 2 \\ -1 & p & 2 \end{pmatrix}$		
(a)	$\det \mathbf{M} = \begin{vmatrix} 3 & 1 & p \\ 1 & 1 & 2 \\ -1 & p & 2 \end{vmatrix}$ $= 3(2-2p) - 1(2+2) + p(p+1)$	Attempts determinant. Requires at least 2 correct "terms". May use other rows/columns or rule of Sarrus.	M1
	$= p^2 - 5p + 2$	Correct simplified determinant.	A1
	$p^2 - 5p + 2 = 0 \Rightarrow p = ...$	Solves 3TQ	M1
	$\dfrac{5 \pm \sqrt{17}}{2}$	Correct values.	A1
			(4)
(b)	Minors $\begin{pmatrix} 2-2p & 4 & p+1 \\ (2-p^2) & 6+p & (3p+1) \\ 2-p & (6-p) & 2 \end{pmatrix}$	Attempts the matrix of minors. If there is any doubt look for at least 6 correct elements. May be implied by their matrix of cofactors.	M1 **(B1 on EPEN)**
	Cofactors $\begin{pmatrix} 2-2p & -4 & p+1 \\ -(2-p^2) & 6+p & -(3p+1) \\ 2-p & -(6-p) & 2 \end{pmatrix}$	Attempts cofactors.	M1
		Correct matrix	A1
	$\mathbf{M}^{-1} = \dfrac{1}{p^2 - 5p + 2} \begin{pmatrix} 2-2p & p^2-2 & 2-p \\ -4 & 6+p & p-6 \\ p+1 & -3p-1 & 2 \end{pmatrix}$	Transposes matrix of cofactors and divides by determinant.	M1
		Follow though their det **M** from part (a) but the adjoint matrix must be correct.	A1ft
			(5)
			Total 9

Question Number	Scheme	Notes	Marks

4(i)	$f(x) = x\arccos x,\ -1 \le x \le 1,$		
	$f'(x) = \arccos x - \dfrac{x}{\sqrt{1-x^2}}$ M1: Differentiates using the product rule to obtain an expression of the form: $\arccos x \pm \dfrac{x}{\sqrt{1-x^2}}$ A1: Correct derivative		M1A1
	$f'(0.5) = \arccos 0.5 - \dfrac{0.5}{\sqrt{1-0.5^2}} = \dfrac{\pi-\sqrt{3}}{3}$	$\dfrac{\pi-\sqrt{3}}{3}$ oe e.g. $\dfrac{\pi}{3}-\dfrac{1}{\sqrt{3}}$	A1
			(3)
(ii)	$g(x) = \arctan(e^{2x})$		
	$g'(x) = \dfrac{2e^{2x}}{e^{4x}+1}$ M1: Differentiates using the chain rule to obtain an expression of the form: $\dfrac{ke^{2x}}{\left(e^{2x}\right)^2+1}$ A1: Correct derivative in any form		M1A1
	$g'(x) = \dfrac{2}{e^{2x}+e^{-2x}} = \text{sech}(2x)$	Introduces sech(2x). Depends on previous M.	**d**M1
	$g''(x) = -2\,\text{sech}(2x)\tanh(2x)$	Differentiates $\text{sech}(u) \to \pm\text{sech}\,u\tanh u$ Depends on both previous M's.	**d**M1
		Correct expression.	A1
			(5)
(ii) **ALT 1**	$g'(x) = \dfrac{2e^{2x}}{e^{4x}+1}$ M1: Differentiates using the chain rule to obtain an expression of the form: $\dfrac{ke^{2x}}{\left(e^{2x}\right)^2+1}$ A1: Correct derivative in any form		M1A1
	$g''(x) = \dfrac{4e^{2x}\left(1+e^{4x}\right)-4e^{4x}\times 2e^{2x}}{(e^{4x}+1)^2}$	Differentiates using quotient or product rule. Depends on first M.	**d**M1
	$= \dfrac{4e^{2x}-4e^{6x}}{(e^{4x}+1)^2} = \dfrac{-4(e^{2x}-e^{-2x})}{(e^{2x}+e^{-2x})^2}$	Multiply through by e^{-4x}. Depends on both previous M's.	**d**M1
	$= -2\dfrac{2}{e^{2x}+e^{-2x}}\dfrac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$ $= -2\,\text{sech}\,2x\tanh 2x$	Correct expression.	A1
	Note that the first derivative may be obtained implicitly in either method e.g. $y = \arctan\left(e^{2x}\right) \Rightarrow \tan y = e^{2x} \Rightarrow \sec^2 y\dfrac{dy}{dx} = 2e^{2x} \Rightarrow \dfrac{dy}{dx} = \dfrac{2e^{2x}}{1+\left(e^{2x}\right)^2}$		
			Total 8

Question Number	Scheme	Notes	Marks
5.	$I_n = \int \sec^n x\,dx, \qquad n \ge 0$		

5(a)	$\int \sec^n x\,dx = \int \sec^{n-2} x \sec^2 x dx$	Splits $\sec^n x$ into $\sec^{n-2} x \sec^2 x$	M1
	$\int \sec^n x\,dx = \sec^{n-2} x \tan x - \int (n-2)\sec^{n-2} x \tan^2 x\,dx$ **Depends on previous M mark** dM1: Uses integration by parts to obtain $\sec^{n-2} x \tan x - k\int \sec^{n-2} x \tan^2 x\,dx$ A1: Correct integration		**d**M1A1
	$\int \sec^n x\,dx = \sec^{n-2} x \tan x - \int (n-2)\sec^{n-2} x(\sec^2 x - 1)\,dx$ Uses $\tan^2 x = \sec^2 x - 1$		B1 **(M1 on EPEN)**
	$\int \sec^n x\,dx = \sec^{n-2} x \tan x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2} x\,dx$ $= \sec^{n-2} x \tan x - (n-2)I_n + (n-2)I_{n-2} \Rightarrow (n-1)I_n = ...$ **Depends on all previous M and B marks** Introduces I_n and I_{n-2} and makes progress to the given result.		**dd**M1
	$(n-1)I_n = \tan x \sec^{n-2} x + (n-2)I_{n-2}$ *	Fully correct proof.	A1cso
			(6)
ALT	$\int \sec^n x\,dx = \int \sec^{n-2} x \sec^2 x dx$	Splits $\sec^n x$ into $\sec^{n-2} x \sec^2 x$	M1
	$\int \sec^{n-2} x \sec^2 x dx = \int \sec^{n-2} x\left(1+\tan^2 x\right)dx$ $= \int \sec^{n-2} x dx + \int \tan^2 x \sec^{n-2} x dx$	Uses $\sec^2 x = 1 + \tan^2 x$ and splits into 2 integrals.	B1 **(4th mark M1 on EPEN)**
	$\int \tan^2 x \sec^{n-2} x dx = \frac{1}{(n-2)}\tan x \sec^{n-2} x - \frac{1}{(n-2)}\int \sec^n x\,dx$ Uses integration by parts on $\int \tan^2 x \sec^{n-2} x dx$ to obtain $A\tan x \sec^{n-2} x - B\int \sec^n x\,dx$ **Note this is the 2nd M on EPEN.**		**d**M1
	$\int \sec^n x\,dx = \int \sec^{n-2} x\,dx + \frac{1}{(n-2)}\tan x \sec^{n-2} x - \frac{1}{(n-2)}\int \sec^n x\,dx$ Fully correct integration		A1
	$\int \sec^n x\,dx = I_{n-2} + \frac{1}{(n-2)}\tan x \sec^{n-2} x - \frac{1}{(n-2)}I_n \Rightarrow (n-1)I_n = ...$ **Depends on previous M and B marks** Introduces I_n and I_{n-2} and makes progress to the given result.		**dd**M1
	$(n-1)I_n = \tan x \sec^{n-2} x + (n-2)I_{n-2}$ *	Fully correct proof.	A1cso

5(b)	$I_2 = 1$	Correct value for I_2 seen or implied.	B1
	$I_6 = \frac{1}{5}\tan x \sec^4 x + \frac{4}{5}I_4$ or e.g.	Applies the given reduction formula once.	M1

	$I_6=\frac{1}{5}\tan\frac{\pi}{4}\sec^4\frac{\pi}{4}+\frac{4}{5}I_4$ or e.g. $I_6=\frac{1}{5}(1)\left(\sqrt{2}\right)^4+\frac{4}{5}I_4$		
	$=\frac{1}{5}\tan x\sec^4 x+\frac{4}{5}\left(\frac{1}{3}\tan x\sec^2 x+\frac{2}{3}I_2\right)=\frac{1}{5}(1)\left(\sqrt{2}\right)^4+\frac{4}{15}(1)\left(\sqrt{2}\right)^2+\frac{8}{15}(1)$ Applies the given reduction formula again and uses the limits to reach a numerical expression for I_6		M1
	$=\frac{28}{15}$	Correct value	A1
			(4)
ALT	$I_2=1$	Correct value for I_2 seen or implied.	B1
	$I_4=\frac{1}{3}\tan x\sec^2 x+\frac{2}{3}I_2$ or e.g. $I_4=\frac{1}{3}\tan\frac{\pi}{4}\sec^2\frac{\pi}{4}+\frac{2}{3}I_2$ or e.g. $I_4=\frac{1}{3}(1)\left(\sqrt{2}\right)^2+\frac{2}{3}I_2$	Applies the given reduction formula once.	M1
	$I_6=\frac{1}{5}\tan x\sec^4 x+\frac{4}{5}\left(\frac{1}{3}\tan x\sec^2 x+\frac{2}{3}I_2\right)=\frac{1}{5}(1)\left(\sqrt{2}\right)^4+\frac{4}{15}(1)\left(\sqrt{2}\right)^2+\frac{8}{15}$ Applies the given reduction formula again and uses the limits to reach a numerical expression for I_6		M1
	$=\frac{28}{15}$	Correct value	A1
			Total 10

In part (b), condone confusion with the coefficients provided the intention is clear.

For either method in part (b), all working must be shown and the given reduction formula must be used at least once. So do not allow e.g. I_4 to be evaluated with a calculator but I_4 can be evaluated directly without using the given reduction formula using an alternative method e.g. by parts or by substitution – see below:

Parts:

$$I_4=\int\sec^4 x\,dx=\int\sec^2 x\sec^2 x\,dx=\sec^2 x\tan x-2\int\sec^2 x\tan^2 x\,dx$$

$$=\sec^2 x\tan x-2\int\sec^2 x\left(\sec^2 x-1\right)dx=\sec^2 x\tan x-2\int\sec^4 x\,dx+2\int\sec^2 x\,dx$$

$$=\sec^2 x\tan x-2I_4+2\int\sec^2 x\,dx\Rightarrow 3I_4=\sec^2 x\tan x+2\tan x\Rightarrow I_4=\frac{1}{3}\sec^2 x\tan x+\frac{2}{3}\tan x$$

Substitution:

$$I_4=\int\sec^4 x\,dx=\int\sec^2 x\sec^2 x\,dx=\int\sec^2 x\left(1+\tan^2 x\right)dx$$

$$u=\tan x\Rightarrow\int\sec^2 x\left(1+\tan^2 x\right)dx=\int\sec^2 x\left(1+u^2\right)\frac{du}{\sec^2 x}=\frac{u^3}{3}+u=\frac{\tan^3 x}{3}+\tan x$$

Question Number	Scheme	Notes	Marks

6(a)	Normal to plane given by $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 3 \\ 1 & -2 & 1 \end{vmatrix} = \ldots$	Attempt cross product of direction vectors. If the method is unclear, look for at least 2 correct components.	M1
	$= 6\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$	Or any multiple of this vector.	A1
	Substitute appropriate point into $6x + 2y - 2z = d$ e.g. (1, 1, 1) or (2, 1, 4) to find "d"	Use a valid point and use scalar product with normal or substitute into Cartesian equation.	M1
	$6x + 2y - 2z = 6$ $3x + y - z = 3$ *	Given answer. No errors seen	A1* cso
			(4)
6(a) ALT	$\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 3\mathbf{k}) + \mu(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ $\Rightarrow x = 1 + \lambda + \mu,\ y = 1 - 2\mu,\ z = 1 + 3\lambda + \mu$ M1: Forms equation of plane using (1, 1, 1) and direction vectors and extracts 3 equations for x, y and z in terms of λ and μ A1: Correct equations		M1A1
	$x = 1 + \frac{1}{2} - \frac{1}{2}y + \frac{1}{3}z - \frac{1}{2} + \frac{1}{6}y$	Eliminates λ and μ and achieves an equation in x, y and z only.	M1
	$3x + y - z = 3$ *	Given answer. No errors seen.	A1
6(b)	$s = -3$	cao	B1
			(1)
6(c)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ 3 & 1 & -1 \end{vmatrix} = \mathbf{i} - 5\mathbf{j} - 2\mathbf{k}$	Attempts cross product of normal vectors. If the method is unclear, look for at least 2 correct components.	M1
	e.g. $x = 0, 2y - 2z = 6, y - 2z = 3$ $\Rightarrow y = 3, z = 0$	Any valid attempt to find a point on the line.	M1
	e.g. (0,3,0)	Any valid point on the line	A1
	$\mathbf{r} = 3\mathbf{j} + \lambda(\mathbf{i} - 5\mathbf{j} - 2\mathbf{k})$	Correct equation including "**r** =" or equivalent e.g. $x = \frac{y-3}{-5} = \frac{z}{-2}$	A1
			(4)
6(c) ALT 1	$\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 3\mathbf{k}) + \mu(\mathbf{i} - 2\mathbf{j} + \mathbf{k}),\ \mathbf{r}.(\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 3$ $\Rightarrow 1 + \lambda + \mu + 1 - 2\mu - 2 - 6\lambda - 2\mu = 3$ Forms equation of first plane using (1, 1, 1) and direction vectors and substitutes into the second plane to form an equation in λ and μ		M1
	$\Rightarrow \mu = \frac{1}{3}(-5\lambda - 3)$	Solves to obtain μ in terms of λ or λ in terms of μ	M1
		Correct equation	A1
	E.g. $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 3\mathbf{k}) + \frac{1}{3}(-5\lambda - 3)(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ Correct equation including "**r** ="		A1
6(c) ALT 2	$3x + y - z = 3,\ x + y - 2z = 3 \Rightarrow 2x + z = 0$	Uses the Cartesian equations of both planes and eliminates one variable	M1
	$z = \lambda \Rightarrow x = -\frac{1}{2}\lambda,\ y = 3 + 2z - x = 3 + \frac{5}{2}\lambda$	Introduces parameter and expresses other 2 variables in terms of the parameter	M1
		Correct equations	A1
	$\mathbf{r} = 3\mathbf{j} + \lambda(\mathbf{i} - 5\mathbf{j} - 2\mathbf{k})$	Correct equation including "**r** =" or equivalent e.g. $x = \frac{y-3}{-5} = \frac{z}{-2}$	A1

6(c) ALT 3	$3x + y - z = 3,\ x + y - 2z = 3 \Rightarrow 2x + z = 0$	Uses the Cartesian equations of both planes and eliminates one variable	M1

	$3x+y-z=3,\ x+y-2z=3 \Rightarrow 5x+y=3$	Uses the Cartesian equations of both planes and eliminates another variable	M1
	$\Rightarrow x=-\frac{z}{2},\quad x=\frac{3-y}{5}$	Correct equations for one variable in terms of the other 2	A1
	$x=\frac{y-3}{-5}=\frac{z}{-2}$	Correct equation or equivalent e.g. $x=\frac{3-y}{5}=\frac{z}{-2}$	A1
6(d)	$(3\mathbf{i}+\mathbf{j}-\mathbf{k})\bullet(\mathbf{i}+\mathbf{j}-2\mathbf{k})=6$	Correct value for scalar product	B1
	$\cos\theta=\frac{(3\mathbf{i}+\mathbf{j}-\mathbf{k}).(\mathbf{i}+\mathbf{j}-2\mathbf{k})}{\sqrt{9+1+1}\sqrt{1+1+4}}=\sqrt{\frac{6}{11}}$	Full scalar product attempt to reach a value for $\cos\theta$	M1
		For $\cos\theta=\sqrt{\frac{6}{11}}$	A1
	$\theta=42.4°$	Correct value. Mark their final answer.	A1
			(4)
6(d) ALT	$\lvert(3\mathbf{i}+\mathbf{j}-\mathbf{k})\times(\mathbf{i}+\mathbf{j}-2\mathbf{k})\rvert=\sqrt{30}$	Correct value for magnitude of cross product	B1
	$\sin\theta=\frac{\lvert(3\mathbf{i}+\mathbf{j}-\mathbf{k}).(\mathbf{i}+\mathbf{j}-2\mathbf{k})\rvert}{\sqrt{9+1+1}\sqrt{1+1+4}}=\frac{\sqrt{55}}{11}$	Full attempt to reach a value for $\sin\theta$	M1
		For $\sin\theta=\frac{\sqrt{55}}{11}$	A1
	$\theta=42.4°$	Correct value. Mark their final answer.	A1
			Total 13

Question Number	Scheme	Notes	Marks
7(i)	$x^2-4x+5=(x-2)^2+1$	Attempts to complete the square. Allow for $(x-2)^2+c,\ c>0$	M1
	$\int \frac{1}{(x-2)^2+1}\mathrm{d}x=\arctan(x-2)$	Allow for $k\arctan\mathrm{f}(x)$.	M1
	$\left[\arctan(x-2)\right]_1^2=0-\left(-\frac{\pi}{4}\right)=\frac{\pi}{4}$	$\frac{\pi}{4}$ cao	A1
			(3)
7(ii)	$\int\frac{\sqrt{x^2-3}}{x^2}\mathrm{d}x=-\frac{\sqrt{x^2-3}}{x}+\int\frac{1}{\sqrt{x^2-3}}\mathrm{d}x$ Uses integration by parts and obtains $A\frac{\sqrt{x^2-3}}{x}+B\int\frac{1}{\sqrt{x^2-3}}\mathrm{d}x$		M1
	$=-\frac{\sqrt{x^2-3}}{x}+\operatorname{arcosh}\frac{x}{\sqrt{3}}$	$B\int\frac{1}{\sqrt{x^2-3}}\mathrm{d}x=k\operatorname{arcosh}\mathrm{f}(x)$	M1
		All correct	A1
	$\int_{\sqrt{3}}^{3}\frac{\sqrt{x^2-3}}{x^2}\mathrm{d}x=\left[-\frac{\sqrt{x^2-3}}{x}+\operatorname{arcosh}\frac{x}{\sqrt{3}}\right]_{\sqrt{3}}^{3}=\left(-\frac{\sqrt{6}}{3}+\operatorname{arcosh}\sqrt{3}\right)-(0+\operatorname{arcosh}1)$ Applies the limits 3 and $\sqrt{3}$ **Depends on both previous M marks**		**d**M1
	$\operatorname{arcosh}\sqrt{3}-\frac{1}{3}\sqrt{6}=\ln(\sqrt{2}+\sqrt{3})-\frac{1}{3}\sqrt{6}$	Accept either of these forms.	A1
			(5)
7(ii) ALT 1	$\int\frac{\sqrt{x^2-3}}{x^2}\mathrm{d}x=\int\frac{\sqrt{3\cosh^2u-3}}{3\cosh^2u}\sqrt{3}\sinh u\,\mathrm{d}u$	A complete substitution using $x=\sqrt{3}\cosh u$	M1
	$=\int\tanh^2u\,\mathrm{d}u$	Obtains $k\int\tanh^2u\,\mathrm{d}u$	M1
	$=\int(1-\operatorname{sech}^2u)\mathrm{d}u=u-\tanh u$	Correct integration	A1
	$\int_{\sqrt{3}}^{3}\frac{\sqrt{x^2-3}}{x^2}\mathrm{d}x=\left[u-\tanh u\right]_0^{\operatorname{arcosh}\sqrt{3}}=\operatorname{arcosh}\sqrt{3}-\tanh\left(\operatorname{arcosh}\sqrt{3}\right)-0$ Applies the limits 0 and arcosh$\sqrt{3}$ **Depends on both previous M marks**		**d**M1
	$\operatorname{arcosh}\sqrt{3}-\frac{1}{3}\sqrt{6}=\ln(\sqrt{2}+\sqrt{3})-\frac{1}{3}\sqrt{6}$	Accept either of these forms.	A1

7(ii) ALT 2	$\int \frac{\sqrt{x^2-3}}{x^2}\mathrm{d}x = \int \frac{\sqrt{3\sec^2 u-3}}{3\sec^2 u}\sqrt{3}\sec u\tan u\mathrm{d}u$	A complete substitution using $x = \sqrt{3}\sec u$	M1
	$= \int \frac{\tan^2 u}{\sec u}\mathrm{d}u$	Obtains $k\int \frac{\tan^2 u}{\sec u}\mathrm{d}u$	M1
	$= \ln(\sec u + \tan u) - \sin u$	Correct integration	A1
	$\int_{\sqrt{3}}^{3} \frac{\sqrt{x^2-3}}{x^2}\mathrm{d}x = \left[\ln(\sec u+\tan u)-\sin u\right]_0^{\operatorname{arcsec}\sqrt{3}}$ $= \ln\left(\sec\left(\operatorname{arcsec}\sqrt{3}\right)+\tan\left(\operatorname{arcsec}\sqrt{3}\right)\right)-\ln\left(\sec(0)+\tan(0)\right)-\sin\left(\operatorname{arcsec}\sqrt{3}\right)$ Applies the limits 0 and arcsec$\sqrt{3}$ **Depends on both previous M marks**		**d**M1
	$\int_{\sqrt{3}}^{3} \frac{\sqrt{x^2-3}}{x^2}\mathrm{d}x = \ln(\sqrt{2}+\sqrt{3})-\frac{1}{3}\sqrt{6}$	Correct answer.	A1
			Total 8

Note that there may be other ways to perform the integration in part (ii) e.g. subsequent substitutions. Marks can be awarded if the method leads to something that is integrable and should be awarded as in the main scheme e.g. M1 for a complete method, M2 for simplifying and reaching an expression that itself can be integrated or can be integrated after rearrangement, A1 for correct integration, dM3 for using appropriate limits and A2 as above.

<u>Alternative approach:</u>

$$\int \frac{\sqrt{x^2-3}}{x^2}\mathrm{d}x = \int \frac{x^2-3}{x^2\sqrt{x^2-3}}\mathrm{d}x = \int \frac{1}{\sqrt{x^2-3}}\mathrm{d}x - \int \frac{3}{x^2\sqrt{x^2-3}}\mathrm{d}x = \operatorname{arcosh}\frac{x}{\sqrt{3}} - ...$$

Can score **M0M1A0dM0A0** if there is no creditable attempt at the second integral.

If the second integral is attempted, it must be using a suitable method e.g. with either $x = \sqrt{3}\cosh u$ or $x = \sqrt{3}\sec u$:

$$\int \frac{3}{x^2\sqrt{x^2-3}}\mathrm{d}x = \int \frac{3}{3\cosh^2 u\sqrt{3\cosh^2 u-3}}\sqrt{3}\sinh u\,\mathrm{d}u = \int \operatorname{sech}^2 u\ \mathrm{d}u = \tanh u + c$$

or

$$\int \frac{3}{x^2\sqrt{x^2-3}}\mathrm{d}x = \int \frac{3}{3\sec^2 u\sqrt{3\sec^2 u-3}}\sqrt{3}\sec u\tan u\mathrm{d}u = \int \cos u\ \mathrm{d}u = \sin u + c$$

In these cases the first M can then be awarded and the other marks as defined with the appropriate limits used.

Question Number	Scheme	Notes	Marks
8(a)	Asymptotes are $y = \pm 2x$	$y = \pm 2x$ oe e.g. $x = \pm \frac{y}{2}$	B1
			(1)
8(b)	$4 = e^2 - 1 \Rightarrow e = \sqrt{5}$	Uses the correct eccentricity formula with $a = 1$ and $b = 2$ to find a value for e.	M1
	Foci are $(\pm\sqrt{5}, 0)$	Both required.	A1
			(2)
8(c)	$8x - 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{4x}{y} = \frac{4\sec\theta}{2\tan\theta}$ or $\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{2\sec^2\theta}{\sec\theta\tan\theta}$ M1: $Ax + By\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = f(\theta)$ or $\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = f(\theta)$ A1: Correct gradient in terms of θ		M1A1
	Explicit differentiation may be seen: $y^2 = 4x^2 - 4 \Rightarrow y = (4x^2 - 4)^{\frac{1}{2}} \Rightarrow \frac{dy}{dx} = \frac{1}{2}(4x^2 - 4)^{-\frac{1}{2}} \times 8x = \frac{4\sec\theta}{\sqrt{4\sec^2\theta - 4}}$ Score M1 for $\frac{dy}{dx} = kx(4x^2 - 4)^{-\frac{1}{2}} = f(\theta)$ and A1 for correct gradient in terms of θ		
	E.g. $y - 2\tan\theta = \frac{4\sec\theta}{2\tan\theta}(x - \sec\theta)$	Correct straight line method using their gradient in terms of θ and $x = \sec\theta$, $y = 2\tan\theta$	M1
	$y\tan\theta - 2\tan^2\theta = 2x\sec\theta - 2\sec^2\theta$ $\Rightarrow y\tan\theta - 2\tan^2\theta = 2x\sec\theta - 2(1 + \tan^2\theta)$		
	$y\tan\theta = 2x\sec\theta - 2$ *	Obtains the given answer with sufficient working shown as above.	A1cso
			(4)
8(d)	$VP : V(-1, 0); P(\sec\theta, 2\tan\theta) \Rightarrow y = \frac{2\tan\theta}{\sec\theta + 1}(x + 1)$ or $WQ : W(1, 0); Q(\sec\theta. -2\tan\theta) \Rightarrow y = \frac{-2\tan\theta}{\sec\theta - 1}(x - 1)$ M1: Correct straight line method for either *VP* or *WQ* A1: One correct equation in any form		M1A1
	$y = \frac{-2\tan\theta}{\sec\theta - 1}(x - 1),\ y = \frac{2\tan\theta}{\sec\theta + 1}(x + 1)$	Both equations correct in any form.	A1
	$\frac{2\tan\theta}{\sec\theta + 1}(x + 1) = \frac{-2\tan\theta}{\sec\theta - 1}(x - 1) \Rightarrow x / y = ...$	Attempt to solve and makes progress to achieve either $x = \ldots$ or $y = \ldots$in terms of θ only.	M1
	$x = \cos\theta$ or $y = 2\sin\theta$	One correct coordinate	A1
	$x = \cos\theta$ and $y = 2\sin\theta$	Both correct	A1
	$x^2 + \frac{y^2}{4} = 1$ or $a = 1, b = 2$	Correct equation or correct values for a and b	A1
			(7)
			Total 14