F3 June 15 IAL M.A Kprime 2

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1. Find the exact values of x for which

$$\cosh 2x - 7\sinh x = 5$$

giving your answers as natural logarithms.

(7)

1. $\cosh 2x = \cosh^2 x + \sinh^2 x$

$c^2 - s^2 = 1$

$\cosh 2x - 7\sinh x = 5$

$\therefore \quad \cosh^2 x + \sinh^2 x - 7\sinh x = 5$

$\therefore \quad 1 + 2\sinh^2 x - 7\sinh x = 5$

$\therefore \quad 2\sinh^2 x - 7\sinh x - 4 = 0$

$(\sinh x - 4)(2\sinh x + 1) = 0$

$\therefore \quad \sinh x = 4 \Rightarrow x = \operatorname{arsinh} 4 = \ln(4 + \sqrt{17})$

$\sinh x = -\frac{1}{2} \Rightarrow x = \ln \frac{\sqrt{5} - 1}{2}$

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2. The hyperbola H has equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where a and b are positive constants.

The hyperbola H has eccentricity $\frac{\sqrt{21}}{4}$ and passes through the point (12, 5).

Find

(a) the value of a and the value of b,

(4)

(b) the coordinates of the foci of H.

(1)

2.(a). @ $(12,5)$, $\frac{144}{a^2} - \frac{25}{b^2} = 1$

Eccentricity $\Rightarrow$ $b^2 = a^2(e^2 - 1)$

$\therefore b^2 = a^2\left(\frac{5}{16}\right)$

$\therefore b^2 = \frac{5}{16}a^2$

$\therefore \frac{144}{a^2} - \frac{25}{\cancel{25} \frac{5}{16}a^2} = 1$

$\therefore \frac{144}{a^2} - \frac{80}{a^2} = 1$

$\therefore a^2 = 64 \Rightarrow a = 8$

$a = 8 \Rightarrow b^2 = 20 \quad \therefore b = 2\sqrt{5}$

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3. $$\mathbf{M} = \begin{pmatrix} 0 & 1 & 9 \\ 1 & 4 & k \\ 1 & 0 & -3 \end{pmatrix}, \text{ where } k \text{ is a constant.}$$

Given that $\begin{pmatrix} 7 \\ 19 \\ 1 \end{pmatrix}$ is an eigenvector of the matrix **M**,

(a) find the eigenvalue of **M** corresponding to $\begin{pmatrix} 7 \\ 19 \\ 1 \end{pmatrix}$, **(2)**

(b) show that $k = -7$ **(2)**

(c) find the other two eigenvalues of the matrix **M**. **(4)**

The image of the vector $\begin{pmatrix} p \\ q \\ r \end{pmatrix}$ under the transformation represented by **M** is $\begin{pmatrix} -6 \\ 21 \\ 5 \end{pmatrix}$.

(d) Find the values of the constants p, q and r. **(4)**

3 (a). $\underline{M}\underline{x} = \lambda \underline{x}$

$\therefore \begin{pmatrix} 0 & 1 & 9 \\ 1 & 4 & k \\ 1 & 0 & -3 \end{pmatrix}\begin{pmatrix} 7 \\ 19 \\ 1 \end{pmatrix} = \begin{matrix} 7\lambda \\ 19\lambda \\ \lambda \end{matrix}$

$\therefore \begin{pmatrix} 28 \\ 83+k \\ 4 \end{pmatrix} = \begin{pmatrix} 7\lambda \\ 19\lambda \\ \lambda \end{pmatrix}$

$\Rightarrow$ $\underline{\lambda = 4}$ is the correspoing E. Value

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Question 3 continued

(b) $\underline{M}\underline{x} = \begin{pmatrix} 28 \\ 83+k \\ 4 \end{pmatrix} = 4\begin{pmatrix} 7 \\ 19 \\ 1 \end{pmatrix} = 4\underline{x}$

$\therefore \quad 83+k = 76$

$\Rightarrow \quad k = 76-83$

$\therefore k = -7$ as required.

(C) $M-\lambda I = \begin{pmatrix} -\lambda & 1 & 9 \\ 1 & 4-\lambda & -7 \\ 1 & 0 & -3-\lambda \end{pmatrix}$

$\det(M-\lambda I)=0 \Rightarrow \cancel{+\lambda(4-\lambda)(3+\lambda)-((4-\lambda)(-3-\lambda)}$

$\Rightarrow \lambda(4-\lambda)(3+\lambda) - (-3-\lambda+7) + 9(0-4+\lambda) = 0$

$\therefore \lambda(4-\lambda)(3+\lambda) - (4-\lambda) - 9(4-\lambda) = 0$

$\therefore \quad (4-\lambda)[\lambda(3+\lambda) - 10] = 0$

$\therefore \quad (4-\lambda)(\lambda^2+3\lambda-10) = 0$

$(4-\lambda)(\lambda-2)(\lambda+5) = 0$

$\therefore \quad \lambda = 2 \ \& \ \lambda = -5$

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Question 3 continued

$$\begin{pmatrix} 0 & 1 & 9 \\ 1 & 4 & -7 \\ 1 & 0 & -3 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} -6 \\ 21 \\ 5 \end{pmatrix}$$

$$\therefore \begin{pmatrix} q + 9r \\ p + 4q - 7r \\ p - 3r \end{pmatrix} = \begin{pmatrix} -6 \\ 21 \\ 5 \end{pmatrix}$$

$\therefore \quad q + 9r = -6$

$\Rightarrow r = \dfrac{-6-q}{9}$

$\therefore \quad \cancel{p - 3r = 5} \qquad p = 5 + 3r = 5 - \dfrac{6+q}{3}$

$p = 3 - \dfrac{q}{3}$

$p + 4q - 7r = 21$

$\therefore \quad 3 - \dfrac{q}{3} + 4q + \dfrac{7}{9}(6+q) = 21$

$\therefore \quad \dfrac{40}{9}q + \dfrac{23}{3} = 21$

$\Rightarrow q = 3$

$\therefore p = 3 - 1 = 2 \quad \therefore r = -1 \quad \left.\begin{matrix} p = 2 \\ q = 3 \\ r = -1 \end{matrix}\right\} //$

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4. $$I_n = \int \cosh^n x \, dx, \quad n \geqslant 0$$

(a) Show that, for $n \geqslant 2$

$$nI_n = \sinh x \cosh^{n-1} x + (n-1)I_{n-2}$$ **(6)**

(b) Hence find the exact value of

$$\int_0^{\ln 2} \cosh^5 x \, dx$$ **(4)**

4(a) $I_n = \int \cosh^n x \, dx = \int \cosh^{n-1} x \cosh x \, dx$

Let $u = \cosh^{n-1} x$ ~~$\therefore u' = (n-1) \cosh$~~

$\therefore u' = \sinh x (n-1) \cosh^{n-2} x$

$v' = \cosh x \quad \therefore v = \sinh x$

$\therefore I_n = \sinh x \cosh^{n-1} x - (n-1) \int \sinh^2 x \cosh^{n-2} x \, dx$

$c^2 - s^2 = 1$

Use $\sinh^2 x = \cosh^2 x - 1$

$\therefore I_n = \sinh x \cosh^{n-1} x - (n-1) \int \cosh^n x - \cosh^{n-2} x \, dx$

$\therefore I_n = \sinh x \cosh^{n-1} x - (n-1)(I_n - I_{n-2})$

$\therefore I_n = \sinh x \cosh^{n-1} x - (n-1) I_n + (n-1) I_{n-2}$

$\therefore n I_n = \sinh x \cosh^{n-1} x + (n-1) I_{n-2}$

as required.

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Question 4 continued

(b) Let $I_n = \int_0^{\ln 2} \cosh^n x \, dx$

~~$n=5 \Rightarrow 5I$ -~~

$\therefore \quad nI_n = \left[\sinh x \cosh^{n-1} x\right]_0^{\ln 2} + (n-1) I_{n-2}$

~~$\therefore nI_n = \frac{3}{4}\cosh^n$~~

$\therefore nI_n = \frac{3}{4}\left(\frac{5}{4}\right)^{n-1} + (n-1) I_{n-2}$

$n=5 \Rightarrow$

$5I_5 = \frac{1875}{1024} + 4I_3$

$n=3 \Rightarrow$

$3I_3 = \frac{75}{64} + 2I_1$

$I_1 = \int_0^{\ln 2} \cosh x \, dx = \left[\sinh x\right]_0^{\ln 2} = \frac{3}{4}$

$\therefore \quad I_3 = \frac{57}{64} \Rightarrow I_5 = \frac{5523}{5120}$ 1.078

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5. The ellipse E has equation $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$

The line L has equation $y = mx + c$, where m and c are constants.

Given that L is a tangent to E,

(a) show that

$$c^2 - 25m^2 = 9$$

(4)

(b) find the equations of the tangents to E which pass through the point (3, 4).

(5)

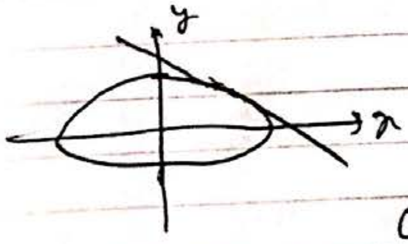
5 (a) $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$ & $y = mx + C$

$\Rightarrow \dfrac{x^2}{25} + \dfrac{(mx+c)^2}{9} = 1$

$\therefore \dfrac{x^2}{25} + \dfrac{m^2x^2 + 2mcx + c^2}{9} = 1$

x225 $\Rightarrow 9x^2 + 25m^2x^2 + 50mcx + 25c^2 = 225$

$\therefore (9 + 25m^2)x^2 + 50mcx + 25c^2 - 225 = 0$



Ellipse & L intersect once iff L is tangent to E.

Quadratic has only one root.

$\therefore$

$\therefore$ discriminant $= 0$

$\Rightarrow 50^2m^2c^2 - 4(9 + 25m^2)(25c^2 - 225) = 0$

$2500m^2c^2 - 4(225c^2 - 2025 + 625m^2c^2 - 5625m^2) = 0$

Question 5 continued

$\therefore\ 2500m^2c^2 - 900c^2 + 8100 - 2500m^2c^2 + 22500\cancel{m}^2 = 0$

$\therefore\ -900c^2 + 22500m^2 + 8100 = 0$

$\downarrow \div (-900)$

$\Rightarrow\ c^2 - 25m^2 - 9 = 0$

$\Rightarrow\ c^2 - 25m^2 = 9$ as required.

(b) $(3,4) \Rightarrow 4 = 3m + c$

$\Rightarrow\ c = 4 - 3m$

$c^2 - 25m^2 = 9$

$\therefore\ (4-3m)^2 - 25m^2 = 9$

$\therefore\ 16 - 24m + 9m^2 - 25m^2 - 9 = 0$

$\therefore\ 16m^2 + 24m - 7 = 0$

$\Rightarrow\ (4m-1)(4m+7) = 0$

$\Rightarrow\ m = \frac{1}{4} \Rightarrow c = 13/4$

$m = -\frac{7}{4} \Rightarrow c = 37/4$

$\therefore\ y = \frac{1}{4}x + \frac{13}{4}$ & $y = -\frac{7}{4}x + \frac{37}{4}$

6.

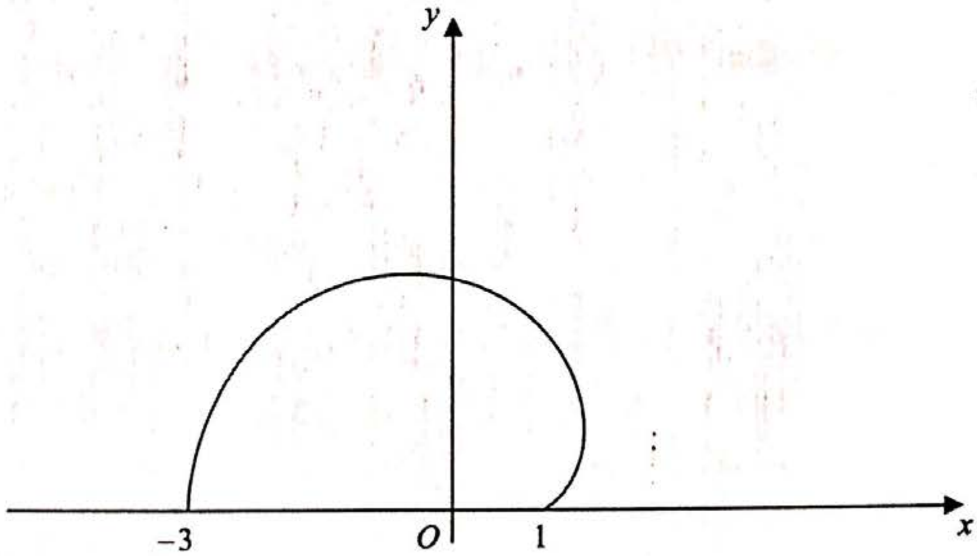


Figure 1

Figure 1 shows the curve C with parametric equations

$$x = 2\cos\theta - \cos 2\theta,\ y = 2\sin\theta - \sin 2\theta, \qquad 0 \leqslant \theta \leqslant \pi$$

(a) Show that

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = 8(1 - \cos\theta)$$

(5)

The curve C is rotated through 2π radians about the x-axis.

(b) Find the area of the surface generated, giving your answer in the form $k\pi$, where k is a rational number.

(5)

$x = 2\cos\theta - \cos 2\theta$

$\therefore \frac{\partial x}{\partial \theta} = -2\sin\theta + 2\sin 2\theta$

$y = 2\sin\theta - \sin 2\theta$

$\frac{\partial y}{\partial \theta} = 2\cos\theta - 2\cos 2\theta$

$$\left(\frac{\partial x}{\partial \theta}\right)^2 = (2\sin 2\theta - 2\sin\theta)^2 = 4\sin^2 2\theta - 8\sin 2\theta \sin\theta + 4\sin^2\theta$$

$$\left(\frac{\partial y}{\partial \theta}\right)^2 = (2\cos\theta - 2\cos 2\theta)^2 = 4\cos^2\theta - 8\cos 2\theta\cos\theta + 4\cos^2 2\theta$$

$$\therefore \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 =$$

$$4\sin^2 2\theta - 8\sin 2\theta\sin\theta + 4\sin^2\theta + 4\cos^2\theta - 8\cos 2\theta\cos\theta + 4\cos^2 2\theta$$

$$= 4(\sin^2 2\theta + \cos^2 2\theta) + 4(\sin^2\theta + \cos^2\theta) - 8\sin 2\theta\sin\theta - 8\cos 2\theta\cos\theta$$

$$\cancel{= 8 - 8(\sin 2\theta \sin\theta}$$

$$\cancel{= 8 + 8(} \quad = 8 - 8(\cos 2\theta\cos\theta + \sin 2\theta\sin\theta)$$

use $\cos(A-B) = \cos A\cos B + \sin A\sin B$

$A = 2\theta \quad B = \theta$

$$\therefore = 8 - 8(\cos(2\theta - \theta)) = 8 - 8\cos\theta$$

$$= 8(1 - \cos\theta)$$

as required.

$2s - 2sc$

$2s(1-c$

$$S = 2\pi \int_0^{\pi} (2\sin\theta - \sin 2\theta)\sqrt{8(1-\cos\theta)}\, d\theta$$

$$= 4\sqrt{2}\,\pi \int_0^{\pi} 2\sin\theta(1-\cos\theta)\sqrt{1-\cos\theta}\, d\theta$$

$$= 8\sqrt{2}\,\pi \int_0^{\pi} \sin\theta\,(1-\cos\theta)^{3/2}\, d\theta$$

$$= 8\sqrt{2}\,\pi \left[\frac{2}{5}(1-\cos\theta)^{5/2}\right]_0^{\pi}$$

$$= 8\sqrt{2}\,\pi \left(\frac{2}{5}(2)^{5/2}\right) \qquad 4\sqrt{2}$$

$$= 8\sqrt{2}\,\pi \left(\frac{2}{5}(\sqrt{2})^5\right)$$

$$= 8\sqrt{2}\,\pi \times \frac{8}{5}\sqrt{2}$$

$$= \frac{128}{5}\pi$$

~~as~~ $k = \frac{128}{5}$

7. The plane Π_1 contains the point $(3, 3, -2)$ and the line $\dfrac{x-1}{2} = \dfrac{y-2}{-1} = \dfrac{z+1}{4}$

(a) Show that a cartesian equation of the plane Π_1 is

$$3x - 10y - 4z = -13$$

(5)

The plane Π_2 is parallel to the plane Π_1

The point $(\alpha, 1, 1)$, where α is a constant, lies in Π_2

Given that the shortest distance between the planes Π_1 and Π_2 is $\dfrac{1}{\sqrt{5}}$

(b) find the possible values of α.

(6)

7(a). $x - 1 = 4 - 2y = \dfrac{2z+2}{4}$

$\therefore\ 4x - 4 = 16 - 8y = 2z + 2$

$4x - 4 = 16 - 8y$

$\therefore\ x = \dfrac{20 - 8y}{4} = 5 - 2y \qquad y = y$

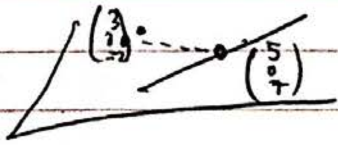
$\cancel{2}\ 2z + 2 = 16 - 8y$

$\therefore\ \cancel{z = 14} \quad z = 7 - 4y$

let $y = \lambda$

line has eqn $\quad \underset{\sim}{r} = \begin{pmatrix} 5 \\ 0 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix}$

$\begin{pmatrix} 5 \\ 0 \\ 7 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 9 \end{pmatrix}$



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$\begin{matrix}2\\-2\end{matrix} \quad \begin{matrix}-3\\1\end{matrix} \times \begin{matrix}9\\-4\end{matrix} \quad \begin{matrix}2\\-2\end{matrix} \times \begin{matrix}-3\\1\end{matrix} \quad \begin{matrix}9\\-4\end{matrix}$

Question 7 continued

$$\therefore \underline{n} = \begin{pmatrix}2\\-3\\9\end{pmatrix} \times \begin{pmatrix}-2\\1\\-4\end{pmatrix}$$

$$= \begin{pmatrix}3\\-10\\-4\end{pmatrix}$$

$$\underline{r}\cdot\underline{n} = \begin{pmatrix}3\\3\\-2\end{pmatrix}\cdot\begin{pmatrix}3\\-10\\-4\end{pmatrix}$$

$$\Rightarrow \underline{r}\cdot\underline{n} = 9 - 30 + 8 = -13$$

$$\therefore \underline{r}\cdot\begin{pmatrix}3\\-10\\-4\end{pmatrix} = -13$$

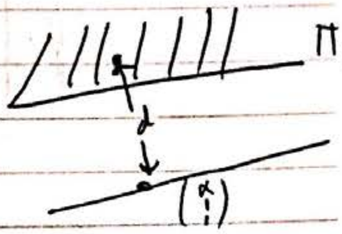
$$\Rightarrow 3x - 10y - 4z = -13$$ as required.

~~(b) Π_2 contains line: $\underline{r} = \begin{pmatrix} \alpha \\ 1 \\ 1 \end{pmatrix} + \mu($~~

(b) Shortest distance = perp. distance from $\begin{pmatrix} \alpha \\ 1 \\ 1 \end{pmatrix}$ to Π_1

Use formula in booklet with

$3x - 10y - 4z + 13 = 0$



$$d = \frac{|3\alpha - 10 - 4 + 13|}{\sqrt{3^2 + 10^2 + 4^2}} = \frac{|3\alpha - 1|}{5\sqrt{5}}$$

$$\Rightarrow \frac{|3\alpha - 1|}{5\sqrt{5}} = \frac{1}{\sqrt{5}}$$

$$\Rightarrow |3\alpha - 1| = 5$$

$\therefore$ $3\alpha - 1 = 5 \Rightarrow \alpha = 2$

$1 - 3\alpha = 5 \Rightarrow \alpha = -\frac{4}{3}$ //

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8. (a) Show that, under the substitution $x = \frac{3}{4}\sinh u$,

$$\int \frac{x^2}{\sqrt{16x^2+9}}\,dx = k\int(\cosh 2u - 1)\,du$$

where k is a constant to be determined.

(6)

(b) Hence show that

$$\int_0^1 \frac{64x^2}{\sqrt{16x^2+9}}\,dx = p + q\ln 3$$

where p and q are rational numbers to be found.

(5)

8(a) $x = \frac{3}{4}\sinh u \quad \therefore \frac{dx}{du} = \frac{3}{4}\cosh u$

$\therefore dx = \frac{3}{4}\cosh u\, du$

$c^2 - s^2 = 1$

$\& \quad \frac{x^2}{\sqrt{16x^2+9}} = \frac{\frac{9}{16}\sinh^2 u}{\sqrt{9\sinh^2 u + 9}} = \frac{9/16 \sinh^2 u}{3\cosh u}$

$= \frac{3}{16}\frac{\sinh^2 u}{\cosh u}$

$\therefore \int \frac{x^2}{\sqrt{16x^2+9}}\,dx = \frac{3}{16}\int \frac{\sinh^2 u}{\cosh u}\cdot\frac{3}{4}\cosh u\, du$

$= \frac{9}{64}\int \sinh^2 u\, du$

Question 8 continued

$$= \frac{9}{128}\int \cosh 2u - 1 \, du$$

$$u = \frac{9}{128}$$

(b) $1 = \frac{3}{4}\sinh u \Rightarrow u = \text{arsinh}\,\frac{4}{3} = \ln 3$

$0 = \sinh u \Rightarrow u = 0$

$$\therefore \frac{9}{2}\int_0^{\ln 3} \cosh 2u - 1 \, du$$

$$= \frac{9}{2}\left[\frac{1}{2}\sinh 2u - u\right]_0^{\ln 3}$$

$$= \frac{9}{2}\left(\frac{1}{2}\sinh \ln 9 - \ln 3\right)$$

$$= \frac{9}{2}\left(\frac{1}{4}(9 - 9^{-1}) - \ln 3\right)$$

5.05

$$= 10 - \frac{9}{2}\ln 3$$