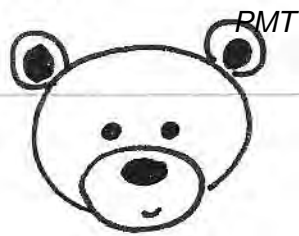


FP2 S15



1. Using algebra, find the set of values of x for which

$$\frac{x}{x+2} < \frac{2}{x+5}$$

(7)

$$\frac{x(x+2)^2(x+5)^2}{\cancel{x+2}} < \frac{2(x+2)^2(x+5)^2}{\cancel{x+5}}$$

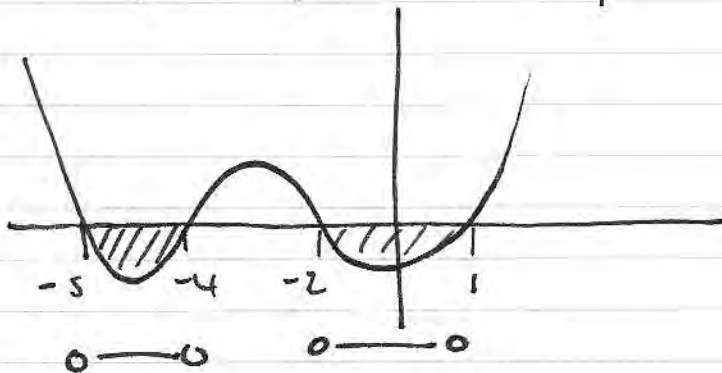
$$\Rightarrow x(x+2)(x+5)^2 - 2(x+2)^2(x+5) < 0$$

$$\Rightarrow (x+2)(x+5) [x(x+5) - 2(x+2)] < 0$$

$$\Rightarrow (x+2)(x+5) [x^2 + 5x - 2x - 4] < 0$$

$$\Rightarrow (x+2)(x+5)(x^2 + 3x - 4) < 0$$

$$\Rightarrow \underset{-2}{(x+2)} \underset{-5}{(x+5)} \underset{-4}{(x+4)} (x-1) < 0$$



$$-5 < x < -4 \quad \text{or} \quad -2 < x < 1$$

2. (a) Express $\frac{1}{(r+6)(r+8)}$ in partial fractions.

(1)

(b) Hence show that

$$\sum_{r=1}^n \frac{2}{(r+6)(r+8)} = \frac{n(an+b)}{56(n+7)(n+8)}$$

where a and b are integers to be found.

(4)

$$\begin{aligned} \text{a) } \frac{1}{(r+6)(r+8)} &= \frac{A}{r+6} + \frac{B}{r+8} \Rightarrow 1 = A(r+8) + B(r+6) \\ r &= -6 \quad 1 = 2A \quad \therefore A = \frac{1}{2} \\ r &= -8 \quad 1 = -2B \quad \therefore B = -\frac{1}{2} \end{aligned}$$

$$= \frac{1}{2(r+6)} - \frac{1}{2(r+8)}$$

$$\text{b) } \frac{2}{(r+6)(r+8)} = 2 \left(\frac{1}{2(r+6)} - \frac{1}{2(r+8)} \right) = \frac{1}{r+6} - \frac{1}{r+8}$$

$$\therefore \sum_{r=1}^n \frac{2}{(r+6)(r+8)} = \sum_{r=1}^n \frac{1}{r+6} - \frac{1}{r+8}$$

$$= \left(\left(\frac{1}{7} - \frac{1}{9} \right) + \left(\frac{1}{8} - \frac{1}{10} \right) + \left(\frac{1}{9} - \frac{1}{11} \right) + \dots \right.$$

$$\left. \left(\frac{1}{n+4} - \frac{1}{n+6} \right) + \left(\frac{1}{n+5} - \frac{1}{n+7} \right) + \left(\frac{1}{n+6} - \frac{1}{n+8} \right) \right)$$

$$= \frac{1}{7} + \frac{1}{8} - \frac{1}{n+7} - \frac{1}{n+8}$$

$$= \frac{15}{56} - \frac{1}{n+7} - \frac{1}{n+8}$$

$$= \frac{15(n+7)(n+8) - 56(n+8) - 56(n+7)}{56(n+7)(n+8)}$$

$$= \frac{15n^2 + 225n + 840 - 56n - 448 - 56n - 392}{56(n+7)(n+8)}$$

$$= \frac{15n^2 + 113n}{56(n+7)(n+8)} = \frac{n(15n + 113)}{56(n+7)(n+8)}$$

3. (a) Show that the substitution $z = y^{-2}$ transforms the differential equation

$$\frac{dy}{dx} + 2xy = xe^{-x^2}y^3 \quad (I)$$

into the differential equation

$$\frac{dz}{dx} - 4xz = -2xe^{-x^2} \quad (II) \quad (4)$$

(b) Solve differential equation (II) to find z as a function of x . (5)

(c) Hence find the general solution of differential equation (I), giving your answer in the form $y^2 = f(x)$. (1)

$$z = y^{-2} \Rightarrow \frac{dz}{dx} = -2y^{-3} \frac{dy}{dx} \quad \therefore \frac{dy}{dx} = -\frac{1}{2}y^3 \frac{dz}{dx}$$

$$\therefore \frac{dy}{dx} + 2xy = xe^{-x^2}y^3 \quad z = \frac{1}{y^2} \Rightarrow y^2 = \frac{1}{z} \Rightarrow y = \frac{1}{z^{1/2}}$$

$$= -\frac{1}{2}y^3 \frac{dz}{dx} + 2xz^{-1/2} = xe^{-x^2}z^{-3/2}$$

$$\frac{-\frac{1}{2}z^{-3/2} \frac{dz}{dx} + 2xz^{-1/2}}{z^{-3/2}} = \frac{xe^{-x^2}z^{-3/2}}{z^{-3/2}}$$

$$-\frac{1}{2} \frac{dz}{dx} + 2xz = xe^{-x^2} \quad (x-2)$$

$$\therefore \frac{dz}{dx} - 4xz = -2xe^{-x^2} \quad \star$$

$$b) \quad IF = e^{-\int 4x dx} = e^{-2x^2}$$

$$e^{-2x^2} \frac{dz}{dx} - 4x e^{-2x^2} z = -2x e^{-x^2} \times e^{-2x^2}$$

$$\frac{d}{dx} [e^{-2x^2} z] = -2x e^{-3x^2}$$

$$e^{-2x^2} z = \int -2x e^{-3x^2} dx$$

$$= \int e^{-3x^2} (-2x dx)$$

$$= \int u \times \frac{1}{3u} du$$

$$\therefore e^{-2x^2} z = \frac{1}{3} u + c$$

$$\therefore e^{-2x^2} z = \frac{1}{3} e^{-3x^2} + c$$

$$\therefore z = \frac{1}{3} e^{-3x^2} \times e^{2x^2} + c e^{2x^2}$$

$$\therefore z = \frac{1}{3} e^{-x^2} + c e^{2x^2}$$

$$u = e^{-3x^2}$$

$$\frac{du}{dx} = -6x e^{-3x^2}$$

$$du = 3e^{-3x^2} (-2x dx)$$

$$\frac{1}{3u} du = -2x dx$$

$$c) \quad \frac{1}{y^2} = \frac{1}{3} e^{x^2} + c e^{2x^2}$$

$$\therefore y^2 = \frac{1}{\frac{1}{3} e^{x^2} + c e^{2x^2}}$$

4. A transformation T from the z -plane to the w -plane is given by

$$w = \frac{z-1}{z+1}, \quad z \neq -1$$

The line in the z -plane with equation $y = 2x$ is mapped by T onto the curve C in the w -plane.

(a) Show that C is a circle and find its centre and radius.

(7)

The region $y < 2x$ in the z -plane is mapped by T onto the region R in the w -plane.

(b) Sketch circle C on an Argand diagram and shade and label region R .

(2)

$$w(z+1) = z-1 \Rightarrow wz + w = z-1 \Rightarrow wz - z = -1-w$$

$$\Rightarrow z - wz = w+1$$

$$\Rightarrow z(1-w) = w+1 \quad \therefore z = \frac{w+1}{1-w} \quad w = u+iv$$

$$\therefore z = \frac{(u+1)+iv}{(1-u)-iv} \times \frac{[(1-u)+iv]}{[(1-u)+iv]} = \frac{(u+1)(1-u)-v^2 + i(v(1-u)+v(u+1))}{(1-u)^2 + v^2}$$

$$\therefore z = \frac{(1-u^2-v^2) + i(v-u+u+v)}{(1-u)^2 + v^2} = x + iy$$

$$\therefore x = \frac{1-u^2-v^2}{(1-u)^2 + v^2} \quad y = \frac{2v}{(1-u)^2 + v^2}$$

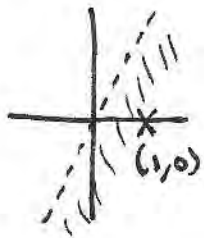
$$y = 2x \quad \therefore \frac{2v}{(1-u)^2 + v^2} = \frac{2(1-u^2-v^2)}{(1-u)^2 + v^2}$$

$$\therefore v = 1-u^2-v^2 \Rightarrow v^2 + v + u^2 = 1$$

$$\Rightarrow \left(v + \frac{1}{2}\right)^2 + u^2 = 1 + \frac{1}{4} = \frac{5}{4} = \left(\frac{\sqrt{5}}{2}\right)^2 \quad \therefore u^2 + \left(v + \frac{1}{2}\right)^2 = \left(\frac{\sqrt{5}}{2}\right)^2$$

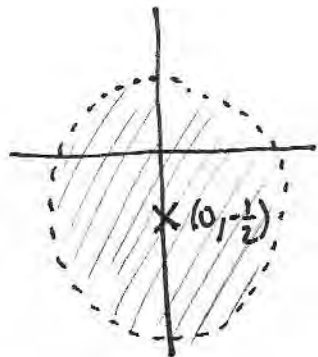
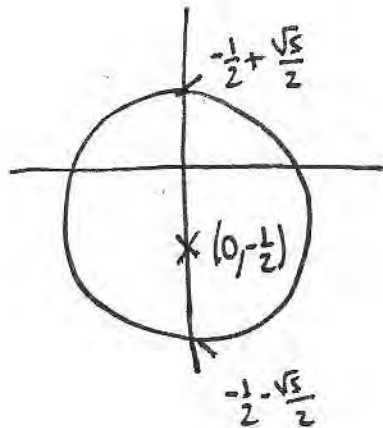
$$C(0, -\frac{1}{2}) \quad r = \frac{\sqrt{5}}{2}$$

$$b) \quad y < 2x$$



$$Z = 1 + 0i$$

$$W = \frac{1 - 1 + 0i}{1 + 1 + 0i} = \frac{0}{2} + 0i \Rightarrow (0, 0)$$



5. Given that $y = \cot x$,

(a) show that

$$\frac{d^2y}{dx^2} = 2 \cot x + 2 \cot^3 x \quad (3)$$

(b) Hence show that

$$\frac{d^3y}{dx^3} = p \cot^4 x + q \cot^2 x + r$$

where p , q and r are integers to be found.

(3)

(c) Find the Taylor series expansion of $\cot x$ in ascending powers of $\left(x - \frac{\pi}{3}\right)$ up to and including the term in $\left(x - \frac{\pi}{3}\right)^3$.

(3)

$$\begin{aligned} \text{a) } y = \cot x &\Rightarrow \frac{dy}{dx} = -\operatorname{cosec}^2 x = -(\operatorname{cosec} x)^2 \\ &\Rightarrow \frac{d^2y}{dx^2} = -2(\operatorname{cosec} x)' x^{-1} \operatorname{cosec} x \cot x \\ &= +2 \operatorname{cosec}^2 x \cot x \end{aligned}$$

$$\frac{\sin^2 + \cos^2}{\sin^2} = \frac{1}{\sin^2} \Rightarrow 1 + \cot^2 = \operatorname{cosec}^2 x$$

$$\begin{aligned} \therefore \frac{d^2y}{dx^2} &= +2 \cot x (1 + \cot^2) \\ &= 2 \cot x + 2 \cot^3 x \end{aligned}$$

$$\begin{aligned} \text{b) } \frac{d^3y}{dx^3} &= -2 \operatorname{cosec}^2 x + 6 (\cot x)^2 x^{-1} \operatorname{cosec}^2 x \\ &= -2 \operatorname{cosec}^2 x (1 + 3 \cot^2 x) \\ &= -2 (1 + \cot^2 x) (1 + 3 \cot^2 x) \\ &= -6 \cot^4 x - 8 \cot^2 x - 2 \end{aligned}$$

$$\left(x - \frac{\pi}{3}\right)$$

$$\cot \frac{\pi}{3} = \frac{1}{\tan \frac{\pi}{3}} = \frac{1}{\sqrt{3}} \quad \cot^2\left(\frac{\pi}{3}\right) = \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{1}{3}$$

$$\operatorname{cosec}\left(\frac{\pi}{3}\right) = \frac{1}{\sin\left(\frac{\pi}{3}\right)} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$$

$$\operatorname{cosec}^2\left(\frac{\pi}{3}\right) = \frac{4}{3}$$

$$\therefore f\left(\frac{\pi}{3}\right) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$f'\left(\frac{\pi}{3}\right) = -\frac{4}{3}$$

$$f''\left(\frac{\pi}{3}\right) = \frac{2}{\sqrt{3}} + 2\left(\frac{1}{\sqrt{3}}\right)^3 = \frac{2}{\sqrt{3}} + \frac{2}{3\sqrt{3}} = \frac{8}{3\sqrt{3}} = \frac{8\sqrt{3}}{9}$$

$$f'''\left(\frac{\pi}{3}\right) = -6\left(\frac{1}{\sqrt{3}}\right)^4 - 8\left(\frac{1}{\sqrt{3}}\right)^2 - 2 = -\frac{6}{9} - \frac{8}{3} - \frac{2}{1} = \frac{-48}{9} = -\frac{16}{3}$$

$$f(x) = \frac{\sqrt{3}}{3} - \frac{4}{3}\left(x - \frac{\pi}{3}\right) + \frac{4}{9}\sqrt{3}\left(x - \frac{\pi}{3}\right)^2 - \frac{8}{9}\left(x - \frac{\pi}{3}\right)^3$$

6. (a) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 2\sin x \quad (I) \tag{8}$$

Given that $y = 0$ and $\frac{dy}{dx} = 1$ when $x = 0$

(b) find the particular solution of differential equation (I). (5)

(CF)

$$y = Ae^{mx}$$

$$y' = Ame^{mx}$$

$$y'' = Am^2e^{mx}$$

$$y'' - 2y' - 3y = 0$$

$$Am^2e^{mx} - 2Ame^{mx} - 3Ae^{mx} = 0$$

$$Ae^{mx}(m^2 - 2m - 3) = 0$$

$$\neq 0 \quad (m-3)(m+1) = 0$$

$$m = 3 \quad m = -1$$

$$\therefore y_{CF} = Ae^{3x} + Be^{-x}$$

(PI)

$$y = P\sin x + Q\cos x$$

$$y' = P\cos x - Q\sin x$$

$$y'' = -P\sin x - Q\cos x$$

$$y'' - 2y' - 3y = 2\sin x$$

$$-P\sin x - Q\cos x$$

$$+ 2Q\sin x - 2P\cos x$$

$$- 3P\sin x - 3Q\cos x$$

$$(-4P + 2Q)\sin x + (-4Q - 2P)\cos x = 2\sin x$$

$$\begin{aligned} -4P + 2Q &= 2 \Rightarrow -4P + 2Q = 2 & \therefore Q = \frac{1}{5} \\ 2P + 4Q &= 0 & 4P + 8Q = 0 \end{aligned}$$

$$2P = -\frac{4}{5} \qquad 10Q = 2 \qquad \therefore P = -\frac{2}{5}$$

$$y_{PI} = -\frac{2}{5}\sin x + \frac{1}{5}\cos x$$

$$\therefore y_{G.S.} = Ae^{3x} + Be^{-x} - \frac{2}{5}\sin x + \frac{1}{5}\cos x$$

$$y = 0 \text{ when } x = 0$$

$$0 = A + B + \frac{1}{5} \Rightarrow 5A + 5B = \text{EMT}$$

$$y' = 1 \text{ when } x = 0$$

$$y' = 3Ae^{3x} - Be^{-x} - \frac{2}{5}\cos x - \frac{1}{5}\sin x$$

$$1 = 3A - B - \frac{2}{5}$$

$$3A - B = \frac{7}{5}$$

$$15A - 5B = 7$$

$$5A + 5B = -1 \quad +$$

$$20A = 6 \quad A = \frac{3}{10}$$

$$\frac{15}{10} = -1 - 5B$$

$$\frac{5}{2} = -5B \quad B = -\frac{1}{2}$$

$$\therefore y = \frac{3}{10}e^{3x} - \frac{1}{2}e^{-x} - \frac{2}{5}\sin x + \frac{1}{5}\cos x$$

7.

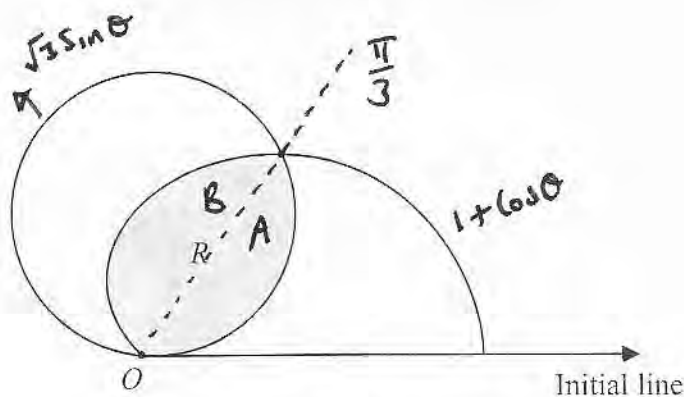


Figure 1

Figure 1 shows the two curves given by the polar equations

$$r = \sqrt{3} \sin \theta, \quad 0 \leq \theta \leq \pi$$

$$r = 1 + \cos \theta, \quad 0 \leq \theta \leq \pi$$

- (a) Verify that the curves intersect at the point P with polar coordinates $\left(\frac{3}{2}, \frac{\pi}{3}\right)$. (2)

The region R , bounded by the two curves, is shown shaded in Figure 1.

- (b) Use calculus to find the exact area of R , giving your answer in the form $a(\pi - \sqrt{3})$, where a is a constant to be found. (6)

$$a) \left(\frac{3}{2}, \frac{\pi}{3}\right) \quad r = \frac{3}{2} \quad \frac{3}{2} = \sqrt{3} \sin \theta \Rightarrow \sin \theta = \frac{3}{2\sqrt{3}}$$

$$\Rightarrow \sin \theta = \frac{3\sqrt{3}}{6} = \frac{\sqrt{3}}{2} \quad \therefore \theta = \frac{\pi}{3} \checkmark$$

$$\text{Area} = \frac{1}{2} \int r^2 d\theta$$

$$\frac{3}{2} = 1 + \cos \theta \Rightarrow \cos \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3} \checkmark$$

$$b) R = A + B \quad A = \int_0^{\frac{\pi}{3}} \frac{1}{2} (\sqrt{3} \sin \theta)^2 d\theta = \frac{3}{2} \int_0^{\frac{\pi}{3}} \sin^2 \theta d\theta$$

$$\cos 2\theta = 1 - 2\sin^2 \theta \quad = \frac{3}{4} \int_0^{\frac{\pi}{3}} 1 - \cos 2\theta d\theta$$

$$\sin^2 \theta = \frac{1}{2} - \frac{1}{2} \cos 2\theta \quad = \frac{3}{4} \left[\theta - \frac{1}{2} \sin 2\theta \right]_0^{\frac{\pi}{3}}$$

$$= \frac{3}{4} \left[\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4} \right) - (0 - 0) \right] = \frac{3\pi}{12} - \frac{3\sqrt{3}}{16} = \frac{\pi}{4} - \frac{3\sqrt{3}}{16}$$

$$B = \int_{\frac{\pi}{3}}^{\pi} \frac{1}{2}(1 + (\cos \theta)^2) d\theta = \frac{1}{2} \int_{\frac{\pi}{3}}^{\pi} (\cos^2 \theta + 2\cos \theta + 1) d\theta$$

$$\cos 2\theta = 2\cos^2 \theta - 1 \Rightarrow \frac{1}{2} \int_{\frac{\pi}{3}}^{\pi} \frac{1}{2} \cos 2\theta + 2\cos \theta + \frac{3}{2} d\theta$$

$$\frac{1}{2} \cos 2\theta + \frac{1}{2}$$

$$\Rightarrow \frac{1}{4} \int_{\frac{\pi}{3}}^{\pi} (\cos 2\theta + 4\cos \theta + 3) d\theta = \frac{1}{4} \left[\frac{1}{2} \sin 2\theta + 4\sin \theta + 3\theta \right]_{\frac{\pi}{3}}^{\pi}$$

$$= \frac{1}{8} \left[\sin 2\theta + 8\sin \theta + 6\theta \right]_{\frac{\pi}{3}}^{\pi}$$

$$= \frac{1}{8} \left[(0 + 0 + 6\pi) - \left(\frac{\sqrt{3}}{2} + 4\sqrt{3} + 2\pi \right) \right]$$

$$= \frac{1}{8} \left[6\pi - 2\pi - \frac{9\sqrt{3}}{2} \right] = \frac{\pi}{2} - \frac{9\sqrt{3}}{16}$$

$$\therefore R = \frac{\pi}{4} + \frac{\pi}{2} - \left(\frac{3\sqrt{3}}{16} + \frac{9\sqrt{3}}{16} \right)$$

$$= \frac{3}{4}\pi - \frac{3}{4}\sqrt{3} = \frac{3}{4}(\pi - \sqrt{3})$$

8. (a) Show that

$$\left(z + \frac{1}{z}\right)^3 \left(z - \frac{1}{z}\right)^3 = z^6 - \frac{1}{z^6} - k \left(z^2 - \frac{1}{z^2}\right)$$

where k is a constant to be found.

(3)

Given that $z = \cos \theta + i \sin \theta$, where θ is real,

(b) show that

(i) $z^n + \frac{1}{z^n} = 2 \cos n\theta$

(ii) $z^n - \frac{1}{z^n} = 2i \sin n\theta$

(3)

(c) Hence show that

$$\cos^3 \theta \sin^3 \theta = \frac{1}{32} (3 \sin 2\theta - \sin 6\theta)$$

(4)

(d) Find the exact value of

$$\int_0^{\frac{\pi}{8}} \cos^3 \theta \sin^3 \theta d\theta$$

(4)

$$\begin{aligned} (z+z^{-1})^3 &= z^3 + 3z^2z^{-1} + 3zz^{-2} + z^{-3} \\ &= z^3 + 3z + 3z^{-1} + z^{-3} \end{aligned}$$

$$(z-z^{-1})^3 = z^3 - 3z + 3z^{-1} - z^{-3}$$

| | | | | | |
|-----------|-----------|------------|------------|------------|--|
| | z^3 | $3z$ | $3z^{-1}$ | z^{-3} | |
| z^3 | z^6 | $3z^4$ | $3z^2$ | z^{-2} | $= z^6 - 3z^2 + 3z^{-2} - z^{-6}$ |
| $-3z$ | $-3z^4$ | $-9z^2$ | -9 | $-3z^{-2}$ | |
| $3z^{-1}$ | $3z^2$ | 9 | $9z^{-2}$ | $3z^{-4}$ | $= z^6 - \frac{1}{z^6} - 3 \left(z^2 - \frac{1}{z^2} \right)$ |
| $-z^{-3}$ | $-z^{-6}$ | $-3z^{-4}$ | $-3z^{-2}$ | $-z^{-6}$ | |

$$b) z = \cos \theta + i \sin \theta$$

$$z^n = \cos n\theta + i \sin n\theta \quad (\text{DEM})$$

$$z^{-n} = \cos(-n\theta) + i \sin(-n\theta)$$

$$z^{-n} = \cos(n\theta) - i \sin(n\theta)$$

$$\underline{z^n = \cos(n\theta) + i \sin(n\theta)} + \quad b) \uparrow -$$

$$z^n + \frac{1}{z^n} = 2 \cos n\theta \quad \text{and} \quad z^n - \frac{1}{z^n} = +2i \sin(n\theta)$$

$$c) z + \frac{1}{z} = 2 \cos \theta \quad z - \frac{1}{z} = -2i \sin \theta$$

$$\begin{aligned} \left(z + \frac{1}{z}\right)^3 \left(z - \frac{1}{z}\right)^3 &= (2 \cos \theta)^3 (+2i \sin \theta)^3 \\ &= 2^3 (\cos \theta)^3 (+2i)^3 (\sin \theta)^3 \\ &= -64i (\cos \theta)^3 (\sin \theta)^3 \end{aligned}$$

$$\therefore -64i (\cos \theta)^3 (\sin \theta)^3 = z^6 - \frac{1}{z^6} - 3 \left(z^2 - \frac{1}{z^2}\right)$$

$$-64i (\cos \theta)^3 (\sin \theta)^3 = (2i \sin 6\theta) - 3(2i \sin 2\theta)$$

$$\div 2i \quad 32 (\cos \theta)^3 (\sin \theta)^3 = -\sin 6\theta + 3 \sin 2\theta$$

$$\therefore \cos^3 \theta \sin^3 \theta = \frac{1}{32} (3 \sin 2\theta - \sin 6\theta) \quad \text{and}$$

$$d) \int_0^{\frac{\pi}{8}} \cos^3 \theta \sin^3 \theta \, d\theta = \frac{1}{32} \int_0^{\frac{\pi}{8}} (3 \sin 2\theta - \sin 6\theta) \, d\theta$$

$$= \frac{1}{32} \left[-\frac{3}{2} \cos 2\theta + \frac{1}{6} \cos 6\theta \right]_0^{\frac{\pi}{8}} = \frac{1}{192} [\cos 6\theta - 9 \cos 2\theta]_0^{\frac{\pi}{8}}$$

$$= \frac{1}{192} \left[\left(\frac{\sqrt{2}}{2} - \frac{9\sqrt{2}}{2}\right) - (1-9) \right] = \frac{1}{192} (8 - 5\sqrt{2})$$