F1 May 16 Model Solutions IAL

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1. Use the standard results for $\sum_{r=1}^{n} r$ and for $\sum_{r=1}^{n} r^3$ to show that, for all positive integers n,

$$\sum_{r=1}^{n} r(r^2-3) = \frac{n}{4}(n+a)(n+b)(n+c)$$

where a, b and c are integers to be found.

(4)

$$\text{LHS} = \sum_{r=1}^{n} r(r^2-3) = \sum_{r=1}^{n} r^3 - 3r$$

$$= \sum_{r=1}^{n} r^3 - 3\sum_{r=1}^{n} r$$

$$= \frac{n^2}{4}(n+1)^2 - \frac{3n}{2}(n+1)$$

$$= \frac{n}{4}(n+1)\left[n(n+1) - 6\right]$$

$$= \frac{n}{4}(n+1)(n^2+n-6)$$

$$= \frac{n}{4}(n+1)(n+3)(n-2)$$

$a = 1$
$b = 3$
$c = -2$

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2. A parabola P has cartesian equation $y^2 = 28x$. The point S is the focus of the parabola P.

(a) Write down the coordinates of the point S.
(1)

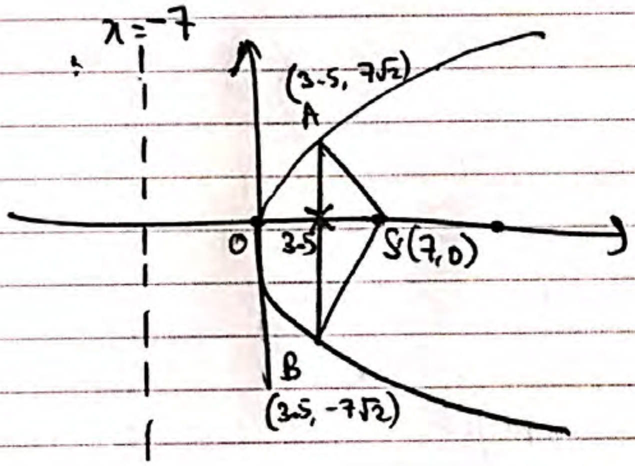
Points A and B lie on the parabola P. The line AB is parallel to the directrix of P and cuts the x-axis at the midpoint of OS, where O is the origin.

(b) Find the exact area of triangle ABS.
(4)

(a) $y^2 = 4 \times 7x$

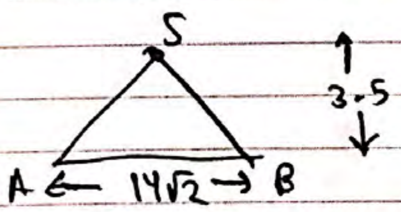
$S: (7, 0)$

(b)



directrix: $x = -7$

$x = 3.5 \Rightarrow y^2 = 98 \Rightarrow y = \pm 7\sqrt{2}$



$\Rightarrow$ Area $= \frac{1}{2} \times 14\sqrt{2} \times 3.5$

$= \dfrac{49\sqrt{2}}{2}$

$\therefore$ Area $\Delta ABS = \dfrac{49\sqrt{2}}{2}$

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$x^2+3x^{-1}-1$

3.

$$f(x) = x^2 + \frac{3}{x} - 1, \qquad x < 0$$

The only real root, α, of the equation $f(x) = 0$ lies in the interval $[-2, -1]$.

(a) Taking -1.5 as a first approximation to α, apply the Newton-Raphson procedure once to $f(x)$ to find a second approximation to α, giving your answer to 2 decimal places. **(5)**

(b) Show that your answer to part (a) gives α correct to 2 decimal places. **(2)**

(a) $f'(x) = 2x - 3x^{-2}$

$f(-1.5) = -\frac{3}{4}$

$f'(-1.5) = -\frac{13}{3}$

$\therefore \alpha \approx -1.5 - \frac{f(-1.5)}{f'(-1.5)} = -\frac{87}{52}$

$\therefore \alpha \approx -1.67$ (2dp)

(b) $f(-1.665) = -0.02957\ldots$

$f(-1.675) = +0.014580\ldots$

$\therefore$ There's a sign change in the interval $[-1.675, -1.665]$

$\Rightarrow -1.675 < \alpha < -1.665$

$\therefore \alpha \approx -1.67$ (2dp)

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4. Given that

$$\mathbf{A} = \begin{pmatrix} k & 3 \\ -1 & k+2 \end{pmatrix}, \text{ where } k \text{ is a constant}$$

(a) show that $\det(\mathbf{A}) > 0$ for all real values of k,

(3)

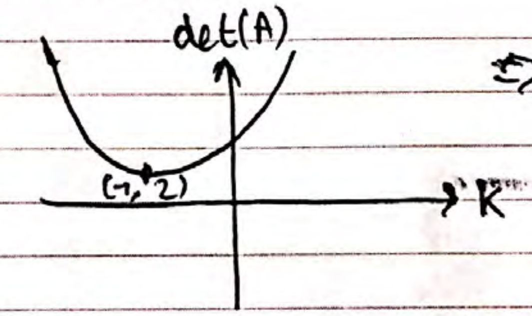
(b) find $\mathbf{A}^{-1}$ in terms of k.

(2)

(a) $\det(A) = k(k+2) + 3 = k^2 + 2k + 3$

$= (k+1)^2 + 2$

$\Rightarrow$



$\Rightarrow \det(A) > 0$ for all $k \in \mathbb{R}$

Also, $\frac{\partial}{\partial k}(k^2+2k+3) = 2k+2$

$\therefore \frac{\partial}{\partial k}[\det(A)] = 2k+2 = 0$

$\Rightarrow k = -1$

$k = -1 \Rightarrow \det(A) = 2 \Rightarrow$ Min point is $(-1, 2)$ $\Big\}$ $\det(A) \geq 2 > 0$

$\therefore \det(A) > 0$ for all k

(b) $A^{-1} = \frac{1}{k^2+2k+3}\begin{pmatrix} k+2 & -3 \\ 1 & k \end{pmatrix}$

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5.

$$2z + z^* = \frac{3 + 4\text{i}}{7 + \text{i}}$$

Find z, giving your answer in the form $a + b\text{i}$, where a and b are real constants. You must show all your working.

(5)

5. $RHS = \frac{3+4i}{7+i} = \frac{(3+4i)(7-i)}{(7+i)(7-i)}$

$= \frac{21+25i-4i^2}{50} = \frac{25+25i}{50}$

$= \frac{1}{2} + \frac{1}{2}i$

$\therefore LHS = 2z+z^* = 2(a+bi) + a-bi$

$= 3a + bi$

$LHS = RHS \Rightarrow 3a + bi = \frac{1}{2} + \frac{1}{2}i$

$\therefore 3a = \frac{1}{2} \Rightarrow a = \frac{1}{6}$

$b = \frac{1}{2}$

$\therefore z = \frac{1}{6} + \frac{1}{2}i$

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6. The rectangular hyperbola H has equation $xy = 25$

(a) Verify that, for $t \neq 0$, the point $P\left(5t, \frac{5}{t}\right)$ is a general point on H. **(1)**

The point A on H has parameter $t = \frac{1}{2}$

(b) Show that the normal to H at the point A has equation

$$8y - 2x - 75 = 0$$

(5)

This normal at A meets H again at the point B.

(c) Find the coordinates of B. **(4)**

6(a) P has parameters $x = 5t$, $y = \frac{5}{t}$

$\therefore x_p y_p = 5t \times \frac{5}{t} = 25$, $t \neq 0$

$\Rightarrow$ For $P(x_p, y_p)$, $xy = 25$

$\therefore$ P lies on H.

(b) $A\left(\frac{5}{2}, 10\right)$

$y = 25x^{-1} \Rightarrow \frac{\partial y}{\partial x} = -25x^{-2} = -\frac{25}{x^2}$

@ A, $\frac{\partial y}{\partial x} = -\frac{25}{(5/2)^2} = -4$

$\Rightarrow$ Gradient of normal $= \frac{1}{4}$

$\therefore y - y_1 = m(x - x_1) \Rightarrow y - 10 = \frac{1}{4}\left(x - \frac{5}{2}\right)$

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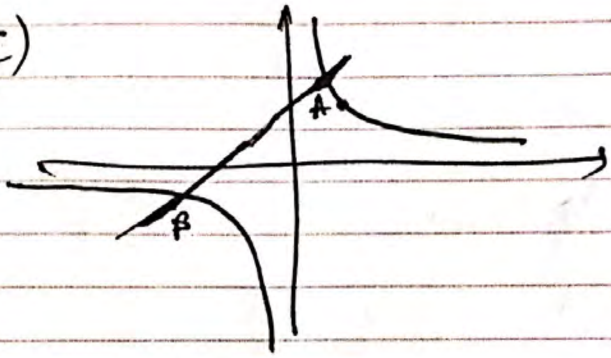
Question 6 continued

$\therefore\ 4y - 40 = x - \frac{5}{2}$

$\times 2 \Rightarrow 8y - 80 = 2x - 5$

$\therefore\ 8y - 2x - 75 = 0$ as required.

(c)



$y = \frac{25}{x} \Rightarrow 8\left(\frac{25}{x}\right) - 2x - 75 = 0$

$\therefore\ \frac{200}{x} - 2x - 75 = 0$

$\times x \Rightarrow 200 - 2x^2 - 75x = 0$

$\therefore\ 2x^2 + 75x - 200 = 0$

$\therefore\ (2x-5)(x+40) = 0$

$x = \frac{5}{2}$ @A $\Rightarrow x_B = -40$

$\therefore\ y_B = \frac{25}{-40} = -\frac{5}{8}$

$\therefore\ B\left(-40, -\frac{5}{8}\right)$

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7.

$$\mathbf{P} = \begin{pmatrix} \frac{5}{13} & -\frac{12}{13} \\ \frac{12}{13} & \frac{5}{13} \end{pmatrix}$$

(a) Describe fully the single geometrical transformation U represented by the matrix **P**. **(3)**

The transformation V, represented by the 2×2 matrix **Q**, is a reflection in the line with equation $y = x$

(b) Write down the matrix **Q**. **(1)**

Given that the transformation $\overset{Q}{V}$ followed by the transformation $\overset{P}{U}$ is the transformation T, which is represented by the matrix **R**,

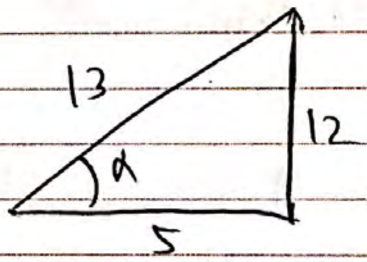
(c) find the matrix **R**. **(2)**

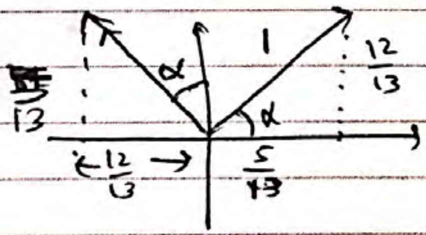
(d) Show that there is a value of k for which the transformation T maps each point on the straight line $y = kx$ onto itself, and state the value of k. **(4)**

(a)

$\tan \alpha = \frac{12}{5}$

$\alpha = 67.4°$ (3sf)





U represents an anticlockwise rotation by $\arctan\left(\frac{12}{5}\right) = 67.4°$ about the origin $(0, 0)$

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Question 7 continued

(b)  $\Rightarrow Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(c) $R = PQ$

$\therefore R = \begin{pmatrix} 5/13 & \frac{-12}{13} \\ 12/13 & 5/13 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} -12/13 & 5/13 \\ 5/13 & 12/13 \end{pmatrix}$

(d) $R \begin{pmatrix} x \\ kx \end{pmatrix} = \begin{pmatrix} x \\ kx \end{pmatrix}$

$R^{-1} = \frac{1}{-1} \begin{pmatrix} 12/13 & -5/13 \\ -5/13 & \frac{-12}{13} \end{pmatrix}$

$\therefore R^{-1} = \begin{pmatrix} -12/13 & 5/13 \\ 5/13 & 12/13 \end{pmatrix} = R$

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Question 7 continued

$\therefore \begin{pmatrix} x \\ kx \end{pmatrix} = R^{-1} \begin{pmatrix} x \\ kx \end{pmatrix}$

$\therefore \begin{pmatrix} x \\ kx \end{pmatrix} = \begin{pmatrix} -12/13 & 5/13 \\ 5/13 & 12/13 \end{pmatrix} \begin{pmatrix} x \\ kx \end{pmatrix}$

$\therefore \begin{pmatrix} x \\ kx \end{pmatrix} = \begin{pmatrix} -\frac{12}{13}x + \frac{5}{13}kx \\ 5/13\, x + \frac{12}{13}kx \end{pmatrix}$

$\therefore \quad x = \cancel{x} \; x\left(\frac{5}{13}k - \frac{12}{13}\right)$

$\Rightarrow \quad \frac{5}{13}k - \frac{12}{13} = 1$

$\Rightarrow k = 5$

$\therefore$ T maps each point ~~onto~~ on $y = kx$ onto itself for $k = 5$

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8.

$$f(z) = z^4 + 6z^3 + 76z^2 + az + b$$

where a and b are real constants.

Given that $-3 + 8i$ is a complex root of the equation $f(z) = 0$

(a) write down another complex root of this equation. **(1)**

(b) Hence, or otherwise, find the other roots of the equation $f(z) = 0$ **(6)**

(c) Show on a single Argand diagram all four roots of the equation $f(z) = 0$ **(2)**

8(a) $-3-8i$

(b) $\left[z-(-3+8i)\right]\left[z-(-3-8i)\right]$

$= z^2 - z(-3-8i) - z(-3+8i) + (-3+8i)(-3-8i)$

$= z^2 + 6z + 9 - 64i^2$

$= z^2 + 6z + 73$ is a factor of f(z)

$\therefore f(z) \equiv (z^2+6z+73)(z^2+uz+v)$

$= z^4 + uz^3 + vz^2 + 6z^3 + 6uz^2 + 6vz$

$+ 73z^2 + 73uz + 73v$

$= z^4 + (u+6)z^3 + \begin{pmatrix} v+6u \\ +73 \end{pmatrix} z^2$

$+ (6v+73u)z + 73v$

compare coeffs: z^3

$\Rightarrow u+6=6 \Rightarrow u=0$

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Question 8 continued

$v = 0 \Rightarrow f(z) = z^4 + 6z^3 + (v + 73)z^2$

$+ 6vz + 73v$

Compare coeffs z^2:

$v + 73 = 76 \Rightarrow v = 3$

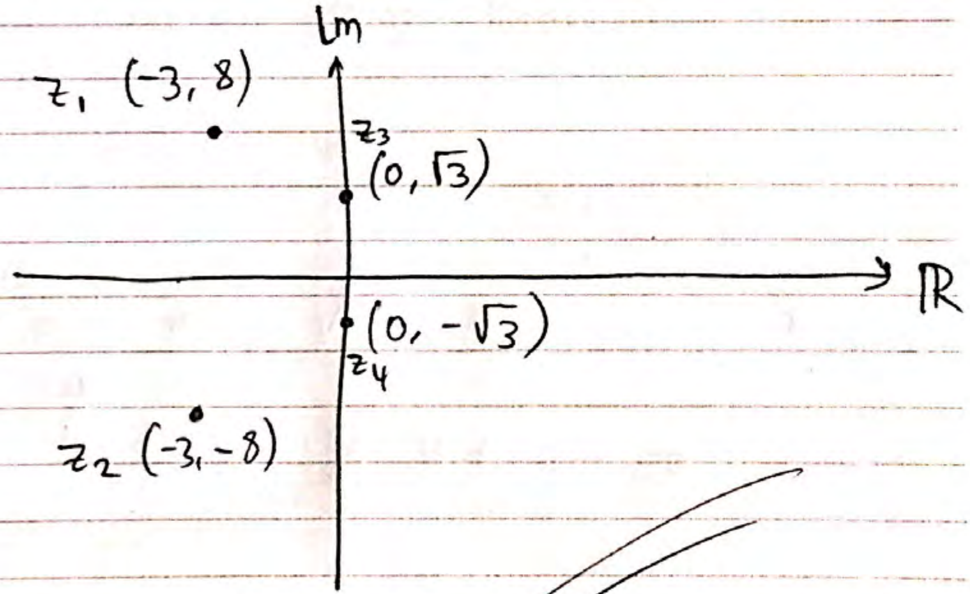
$\therefore f(z) = z^4 + 6z^3 + 76z^2 + 18z + 219$

$\Rightarrow a = 18$
$b = 219$

$z^2 + 3 = 0 \Rightarrow z = \pm i\sqrt{3}$

(c) $z^2 + 3$ is another factor

$\Rightarrow z = \pm i\sqrt{3}$



Q8

(Total 9 marks)

9. The quadratic equation

$$2x^2 + 4x - 3 = 0$$

has roots α and β.

Without solving the quadratic equation,

(a) find the exact value of

(i) $\alpha^2 + \beta^2$

(ii) $\alpha^3 + \beta^3$

(5)

(b) Find a quadratic equation which has roots $(\alpha^2 + \beta)$ and $(\beta^2 + \alpha)$, giving your answer in the form $ax^2 + bx + c = 0$, where a, b and c are integers.

(4)

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9(a)(i) $(x-\alpha)(x-\beta) = \frac{2x^2+4x-3}{2} = x^2+2x-\frac{3}{2}$

$\therefore x^2 - \beta x - \alpha x + \alpha\beta = x^2+2x-3/2$

$\therefore x^2-(\alpha+\beta)x+\alpha\beta = x^2+2x-3/2$

$\therefore (\alpha+\beta) = -2$

$\alpha\beta = -3/2$

$(\alpha+\beta)^2 = \alpha^2+\beta^2+2\alpha\beta$

$\therefore \alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$\therefore \alpha^2+\beta^2 = 4-2(-\frac{3}{2}) = 4+3 = 7$

$\therefore \alpha^2+\beta^2 = 7$

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Question 4 continued

(ii) $(\alpha+\beta)^3 = \alpha^3+\beta^3+3\alpha\beta^2+3\beta\alpha^2$

$= \alpha^3+\beta^3+3\alpha\beta(\alpha+\beta)$

$\therefore \alpha^3+\beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$

$\therefore \alpha^3+\beta^3 = -8-9=-17$

(b) $[x-(\alpha^2+\beta)][x-(\beta^2+\alpha)]$

$= x^2 - x(\beta^2+\alpha) - x(\alpha^2+\beta) + (\alpha^2+\beta)(\beta^2+\alpha)$

$= x^2 - (\alpha^2+\beta^2+\alpha+\beta)x + \alpha^2\beta^2 + \alpha^3+\beta^3+\alpha\beta$

$= x^2-(7-2)x + \frac{9}{4} - 17 - 3/2$

$= x^2 - 5x - \frac{65}{4} = 0$

$\therefore \quad 4x^2 - 20x - 65 = 0$

$a = 4$

$b = -20$

$c = -65$

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10. (i) A sequence of positive numbers is defined by

$$u_1 = 5$$
$$u_{n+1} = 3u_n + 2, \qquad n \geqslant 1$$

Prove by induction that, for $n \in \mathbb{Z}^+$,

$$u_n = 2 \times (3)^n - 1$$

(5)

(ii) Prove by induction that, for $n \in \mathbb{Z}^+$,

$$\sum_{r=1}^{n} \frac{4r}{3^r} = 3 - \frac{(3 + 2n)}{3^n}$$

(6)

10(i). When $n=1$, $u_1 = 2 \times (3)^1 - 1 = 6 - 1 = 5$

$\therefore$ $u_1 = 5$ is correct as given.

$\therefore$ result is true for $n=1$,

When $n=k$, let us assume,

$u_k = 2 \times (3)^k - 1$ is true.

Also, $n=k \Rightarrow u_{k+1} = 3u_k + 2$

$\therefore$ u_{k+1} is given by ...

$$u_{k+1} = 3[2 \times (3)^k - 1] + 2$$
$$= 6(3)^k - 1$$
$$= 2 \cdot 3 \cdot 3^k - 1 = 2(3)^{k+1} - 1$$

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Question 10 continued

$= 2\times(3)^{k+1} - 1$ as it should for $n=k+1$ under $u_n = 2\times(3)^n - 1$

$\therefore$ Result is shown true for $n=k+1$

If the result is true for $n=k$, it is shown to be true for $n=k+1$. Since it's true for $n=1$ it must be true for $n=2,3,4,5\ldots$ and all $n\in\mathbb{Z}^+$ by induction.

(ii) When $n=1$, LHS$= \sum_{r=1}^{1}\frac{4r}{3^r} = \frac{4\times1}{3^1} = \frac{4}{3}$

RHS$= 3 - \frac{3+2}{3} = 3 - \frac{5}{3} = \frac{4}{3}$

$\therefore$ LHS = RHS is true for $n=1$.

$\therefore$ result is true for $n=1$.

~~When~~, let's assume for $n=k$,

LHS $= \sum_{r=1}^{k}\frac{4r}{3^r} = 3 - \frac{(3+2k)}{3^k}$ = RHS

is true.

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Question 10 continued

When $n=k+1$,

We must show

$$\cancel{LHS}\ \sum_{r=1}^{k+1} \frac{4r}{3^r} = 3 - \frac{[3+2(k+1)]}{3^{k+1}}$$

$$= 3 - \frac{5+2k}{3^{k+1}}.$$

Now when $n=k+1$

$$LHS: \sum_{r=1}^{k+1} \frac{4r}{3^r} = \sum_{r=1}^{k} \frac{4r}{3^r} + \frac{4(k+1)}{3^{k+1}}$$

$$= 3 - \frac{(3+2k)}{3^k} + \frac{4(k+1)}{3^{k+1}}$$

$$= 3 - \frac{3(3+2k)}{3^{k+1}} + \frac{4(k+1)}{3^{k+1}}$$

$$= 3 + \frac{4k+4-9-6k}{3^{k+1}} = 3 + \frac{-2k-5}{3^{k+1}}$$

$$= 3 - \frac{5+2k}{3^{k+1}}$$

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Question 10 continued

$$= 3 - \frac{3+2(k+1)}{3^{k+1}}$$ as it should

$\therefore$ Result is true for $n=k+1$

If the result is true for $n=k$, it is shown true for $n=k+1$, since it's true for $n=1$, it must be true for $n=2,3,4,\ldots$ and all $n \in \mathbb{Z}^+$ by induction.

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